B5.4 Waves & Compressible Flow

Question Sheet 4 Solutions

1. The derivation of the equations is in the lecture notes, as is the fact that they can be rewritten as

$$\left(\frac{\partial}{\partial t} + (u \pm c)\frac{\partial}{\partial x}\right)(u \pm 2c) = 0, \tag{1}$$

were $c = \sqrt{h}$, and therefore $u \pm 2c$ is constant on characteristic curves $\mathrm{d}x/\mathrm{d}t = u + c$.

Pressure is $p = \rho g(h - z)$ so force on the dam is $F = \int_0^h p\,\mathrm{d}z = \frac{1}{2}\rho g h^2$. If flux is proportional to force, $uh \propto \frac{1}{2}\rho g h^2$, implying $u \propto -gh = -c^2$, where $c = \sqrt{gh}$ (the sign is because u must be negative). Thus, we can write

$$u = -kc^2, \tag{2}$$

for some $k > 0$.

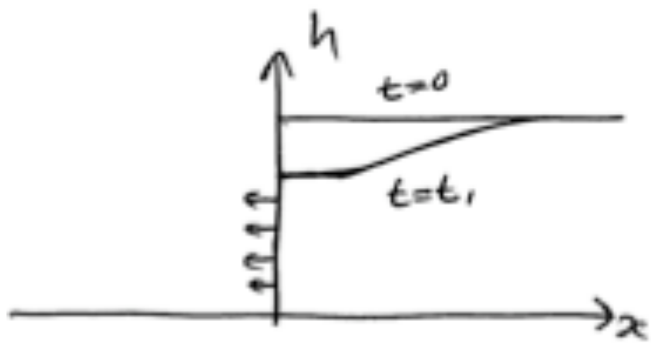

Figure 1: Question 1

Recall that $u \pm 2c$ are constant along the characteristic curves $dx/dt = u \pm c$. For boundary and initial conditions, we have $u = 0$, $h = h_0$ (hence $c = c_0$) on $\{t = 0,\, x > 0\}$, and $u = -kc^2$ on $\{x = 0,\, t > 0\}$.

Characteristics that come from $\{t = 0,\, x > 0\}$ have

$$u + 2c = 2c_0, \tag{3}$$

$$u - 2c = -2c_0, \tag{4}$$

along them, so where such characteristics intersect (and both of these equations therefore hold), we have $u = 0$, $c = c_0$. On that region, these characteristics are therefore $dx/dt = \pm c_0$, *i.e* straight lines (region I).

One set of characteristics that come from the dam at $\{x = 0,\, t > 0\}$ have negative slope so pass out of the domain and are unimportant (the 'negative' characteristics). The other 'positive' characteristics that come from $\{x = 0,\, t > 0\}$ have

$$u + 2c = -kc_*^2 + 2c_*, \tag{5}$$

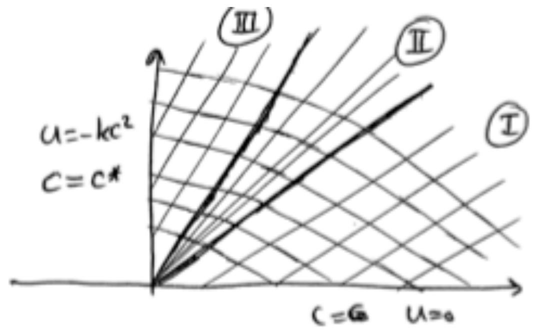

Figure 2: Characteristic diagram for Question 1

where c_* (potentially dependent on t) is the value of c at the dam $x = 0$. Where these intersect with the negative characteristics from $\{t = 0,\ x > 0\}$, we also have

$$u - 2c = -2c_0, \tag{6}$$

and the combination of these two equations implies that u and c are constant along each of these positive characteristics. Therefore they are straight lines. The condition at $x = 0$ also gives

$$-2c_0 + 4c_* = -kc_*^2 + 2c_* \quad \Longrightarrow \quad c_* = \frac{-1 + (1 + 2c_0k)^{1/2}}{k}, \tag{7}$$

and hence, on all the region where these characteristics reach (region III), we have

$$c = c_* = \frac{-1 + (1 + 2c_0k)^{1/2}}{k}, \qquad u = -\frac{2}{k}\left(1 + c_0k - (1 + 2c_0k)^{1/2}\right). \tag{8}$$

Finally, on the remaining region (region II) there must be an expansion fan with positive characteristics emanating from the origin. On each of these $u + 2c$ is constant, and together with the condition $u - 2c = -2c_0$ from the negative characteristics, this still implies that u and c are constant along each positive characteristic. Therefore they are straight lines, given by

$$\frac{x}{t} = u + c. \tag{9}$$

Combining this with the invariant from the negative characteristics, $u - 2c = -2c_0$, we deduce

$$c = \frac{2c_0}{3} + \frac{1}{3}\frac{x}{t}, \qquad u = \frac{2}{3}\frac{x}{t} - \frac{2}{3}c_0, \tag{10}$$

on this region.

The dividing characteristics that separate the different regions are given by those on which $c = c_*$ and $c = c_0$, so

$$\frac{x}{t} = 3c_* - 2c_0 = c_0 - \frac{3}{k}\left(1 + c_0k - (1 + 2c_0k)^{1/2}\right), \tag{11}$$

and

$$\frac{x}{t} = c_0. \tag{12}$$

2. (a) Writing $u = cM$, the Rankine-Hugoniot conditions are

$$[\rho M c]_-^+ = \left[p + \rho c^2 M^2\right]_-^+ = \left[\frac{c^2 M^2}{2} + \frac{\gamma p}{(\gamma-1)\rho}\right]_-^+ = 0. \tag{1}$$

With $c^2 = \gamma p/\rho$, these give

$$\left[\gamma M^2 \rho\, p\right]_-^+ = 0 \quad \Longrightarrow \quad \left(\frac{p_+}{p_-}\right)\left(\frac{\rho_+}{\rho_-}\right) = \frac{M_-^2}{M_+^2}, \tag{2}$$

$$\left[p(1+\gamma M^2)\right]_-^+ = 0 \quad \Longrightarrow \quad \frac{p_+}{p_-} = \frac{1+\gamma M_-^2}{1+\gamma M_+^2}, \tag{3}$$

$$\left[\frac{\gamma p}{\rho}\left(M^2 + \frac{2}{\gamma-1}\right)\right]_-^+ = 0 \quad \Longrightarrow \quad \left(\frac{p_+}{p_-}\right)\left(\frac{\rho_-}{\rho_+}\right) = \frac{2+(\gamma-1)M_-^2}{2+(\gamma-1)M_+^2}. \tag{4}$$

To find a relationship between the Mach numbers, eliminate the ratios of pressures and densities to give

$$\frac{M_-^2}{M_+^2}\left(\frac{2+(\gamma-1)M_-^2}{2+(\gamma-1)M_+^2}\right)\left(\frac{1+\gamma M_+^2}{1+\gamma M_-^2}\right)^2 = 1. \tag{5}$$

This has the obvious solution $M_-^2 = M_+^2$, so factorising that gives

$$(M_-^2 - M_+^2)\left[2 + (\gamma-1)(M_-^2 + M_+^2) - 2\gamma M_-^2 M_+^2\right] = 0, \tag{6}$$

and since $M_-^2 \neq M_+^2$ for there to be a shock,

$$M_+^2 = \frac{2+(\gamma-1)M_-^2}{2\gamma M_-^2 - (\gamma-1)}. \tag{7}$$

[Note that this can also be written as

$$M_+^2 = \frac{\gamma-1}{2\gamma} + \frac{1}{2\gamma}\frac{(\gamma+1)^2}{(\gamma+1) + 2\gamma(M_-^2-1)}, \tag{8}$$

so M_+ is a decreasing function of M_-, with $M_+ = 1$ when $M_- = 1$. Thus one of them must always be larger than 1 and the other one less than 1.]

Substituting this into the ratio of pressures and rearranging gives

$$\frac{p_+}{p_-} = 1 + \frac{2\gamma}{\gamma+1}(M_-^2 - 1), \tag{9}$$

and then

$$\frac{\rho_+}{\rho_-} = \frac{M_-^2}{M_+^2}\frac{p_-}{p_+} = \frac{(\gamma+1)M_-^2}{2+(\gamma-1)M_-^2}, \tag{10}$$

or

$$\frac{\rho_-}{\rho_+} = 1 - \frac{2(M_-^2-1)}{(\gamma+1)M_-^2}. \tag{11}$$

Now consider

$$E(M_-^2) = \frac{p_+}{\rho_+^\gamma}\bigg/\frac{p_-}{\rho_-^\gamma} = \left(1 + \frac{2\gamma}{\gamma+1}(M_-^2-1)\right)\left(1 - \frac{2(M_-^2-1)}{(\gamma+1)M_-^2}\right)^\gamma. \tag{12}$$

Clearly $E(1) = 1$. We want to show $E(M_-^2)$ is monotonic increasing since then $E(M_-^2) > 1$ if and only if $M_-^2 > 1$ (in which case, also, $M_+^2 < 1$). i.e. the flow must change from supersonic to subsonic as the gas crosses the shock.

To show the monotonicity of $E(M_-^2)$, application of the product rule and algebraic manipulations give

$$E'(M_-^2) = \frac{2\gamma}{\gamma+1}\left(1 - \frac{2(M_-^2-1)}{(\gamma+1)M_-^2}\right)^{\gamma-1}\left[\left(1 - \frac{2(M_-^2-1)}{(\gamma+1)M_-^2}\right) - \frac{1}{M_-^4}\left(1 + \frac{2\gamma}{(\gamma+1)}(M_-^2-1)\right)\right] \tag{13}$$

$$= \frac{2\gamma}{(\gamma+1)^2 M_-^4}\left(1 - \frac{2(M_-^2-1)}{(\gamma+1)M_-^2}\right)^{\gamma-1}\left[(M_-^2+1)(\gamma+1) - 2M_-^2 - 2\gamma\right](M_-^2-1) \tag{14}$$

$$= \frac{2\gamma(\gamma-1)}{(\gamma+1)^2 M_-^4}\left(1 - \frac{2(M_-^2-1)}{(\gamma+1)M_-^2}\right)^{\gamma-1}(M_-^2-1)^2 \tag{15}$$

$$> 0, \tag{16}$$

except at $M_-^2 = 1$ (in that case there is no shock, since $M_- = M_+$).

Finally, the ideal gas law $p = \rho RT$ gives

$$\frac{T_+}{T_-} = \frac{p_+}{p_-}\frac{\rho_-}{\rho_+} \tag{17}$$

$$= E(M_-)^2\left(\frac{\rho_-}{\rho_+}\right)^{1-\gamma} \tag{18}$$

$$= E(M_-)^2\left(\frac{1}{1 - \frac{2(M_-^2-1)}{(\gamma+1)M_-^2}}\right)^{\gamma-1}. \tag{19}$$

$$\tag{20}$$

If $M_-^2 > 1$, both of these terms are larger than 1, and if $M_-^2 < 1$ they are both less than 1. Hence $T_+/T_- > 1$ if and only if $M_-^2 > 1$.

(b) We work in a frame in which the shock is stationary, and which therefore moves backwards from the end of the tube with the shock speed V. In that frame we have $u_- = U + V$, $p_- = p_0$, $\rho_- = \rho_0$, while $u_+ = V$.

The first two shock conditions give

$$\rho_+ = \frac{\rho_- u_-}{u_+}, \tag{21}$$

$$p_+ = p_- + \rho_- u_-^2 - \rho_- u_+ u_-, \tag{22}$$

and substituting into the final condition and rearranging gives

$$\tfrac{1}{2}(u_+ - u_-)(u_+ + u_-)u_- + \frac{\gamma p_-}{(\gamma-1)\rho_-}(u_+ - u_-) + \frac{\gamma}{\gamma-1}u_- u_+(u_- - u_+) = 0. \tag{23}$$

Discounting the spurious solution $u_- = u_+$, which would require $U = 0$, this gives

$$\tfrac{1}{2}(u_+ + u_-)u_- + \frac{c_0^2}{\gamma-1} - \frac{\gamma}{\gamma-1}u_- u_+ = 0. \tag{24}$$

and on substituting $u_- = U + V$, $u_+ = V$ and rearranging, we obtain

$$2(U+V)^2 - (\gamma+1)U(U+V) - 2c_0^2 = 0. \tag{25}$$

This has solution

$$U + V = \frac{(\gamma+1)U + \sqrt{(\gamma+1)^2U^2 + 16c_0^2}}{4}, \tag{26}$$

(positive root since $U + V$ must be positive), and hence

$$V = \frac{\sqrt{(\gamma+1)^2U^2 + 16c_0^2} - (3-\gamma)U+}{4}. \tag{27}$$

3. (a) See lecture notes.

(b) The first condition shows that $q = h_-u_- = h_+u_+$. Substituting $u_\pm = q/h_\pm$ into the second condition gives

$$\left[\frac{q^2}{h} + \frac{gh^2}{2}\right]_-^+ = 0, \tag{1}$$

which indicates that the function $f(h) = q^2/h + gh^2/2$, must take the same value at $h = h_-$ and $h = h_+$. Considering the graph of this function (sketch it) there is a minimum at $gh = q^2/h^2 \implies h = q^{2/3}/g^{1/3}$, and the function grows to infinity to either side of this minimum. Thus for any value of h_-, there is a unique different value of h_+ at which $f(h)$ takes the same value. What is more, if one of these values is to the right of the minimum, the other must be to the left, and vice versa.

(c) See lecture notes.

(d) Move to a frame in which the bore is stationary and then the same conditions apply in that frame. So replace $u_\pm$ with $u_\pm - V$.

(e) Shock conditions for the moving bore are

$$[h(u-V)]_-^+ = 0, \qquad \left[h(u-V)^2 + \tfrac{1}{2}gh^2\right]_-^+ = 0, \tag{2}$$

and we have $u_+ = 0$. Hence

$$-h_+V = h_-(u_- - V), \tag{3}$$

and substituting for $(u_- - V)$ in the momentum condition gives

$$h_+V^2 + \tfrac{1}{2}gh_+^2 = \frac{h_+^2V^2}{h_-} + \tfrac{1}{2}gh_-^2, \tag{4}$$

which rearranges to give (factorizing $(h_- - h_+) \neq 0$),

$$V^2 = \frac{g(h_- + h_+)h_-}{2h_+}. \tag{5}$$

V should be positive, since the fluid depth must increase as fluid flows through the bore.

4. Conserving mass flux into / out of the shock requires

$$[\rho u]_-^+ = 0. \tag{1}$$

Momentum is $\rho\mathbf{u}$, so momentum flux into / out of the shock is $\rho\mathbf{u}u$, where $\mathbf{u} = (u, v)$. Any change in momentum flux is due to the force on the shock, which is due to the pressure jump $-[p]_-^+$ and is oriented only in the x direction. Thus

$$\left[p + \rho u^2\right]_-^+ = 0, \qquad [\rho uv]_-^+ = 0. \tag{2}$$

Combined with the mass condition that ρu is constant across the shock, the latter indicates that $v_- = v_+$ is conserved.

Finally, the energy is given by $e = \frac{1}{2}\rho(u^2+v^2) + \rho c_v T$. Any change in the energy flux eu through the shock is due to the rate of work done by the pressure force $-[pu]_-^+$. Thus

$$\left[\rho u\left(\tfrac{1}{2}(u^2+v^2) + c_v T + \frac{p}{\rho}\right)\right]_-^+ = 0, \tag{3}$$

Using the ideal gas law to write $c_v T = p/(\gamma-1)\rho$, and using $[\rho u]_-^+ = 0$ and $[v]_-^+ = 0$, this gives

$$\left[\tfrac{1}{2}u^2 + \frac{\gamma p}{(\gamma-1)\rho}\right]_-^+ = 0. \tag{4}$$

The second law of thermodynamics states that entropy is non-decreasing. This condition requires that the jump in entropy must be positive as the flow passes through the shock and by looking at that jump one may show that $\rho_+ > \rho_-$ (see the question above, where it was shown that $M_- > 1 > M_+$, so $u_- > c_0 > u_+$, and hence $\rho_+ > \rho_-$).

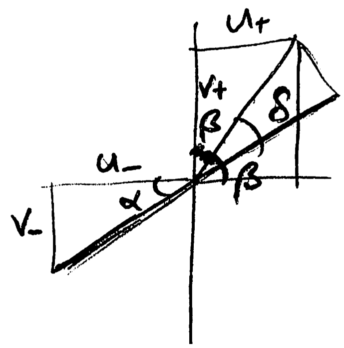

Figure 3: Question 4

Let the initial angle of the shock relative to the normal be α (the angle of incidence), and the final angle be β, with $\beta = \alpha + \delta$ where δ is the deflection. Then

$$\tan\alpha = \frac{v_-}{u_-}, \qquad \tan\beta = \frac{v_+}{u_+}, \tag{5}$$

and so

$$\tan\delta = \frac{\tan\beta - \tan\alpha}{1 + \tan\alpha\tan\beta} = \frac{u_- - u_+}{v_- + \frac{u_+u_-}{v_-}}. \tag{6}$$

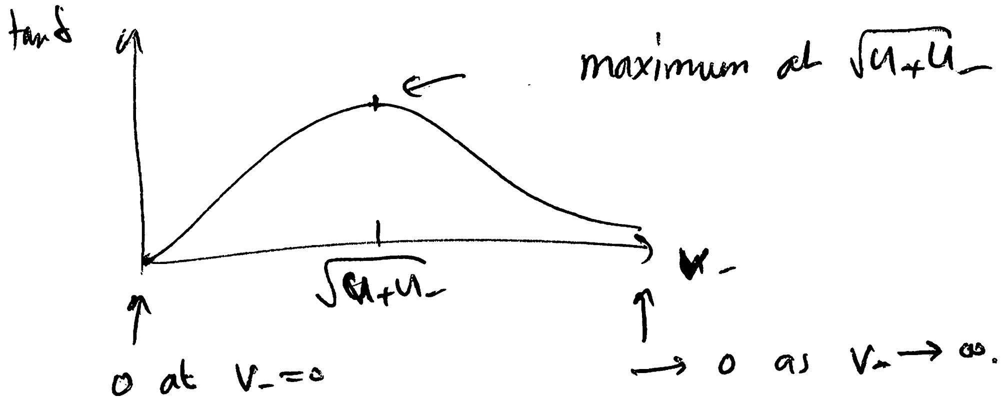

Figure 4: Question 4

Differentiate with respect to v_- to find stationary point at $v_-^2 = u_+u_-$, at which

$$\tan\delta = \frac{u_- - u_+}{2\sqrt{u_+u_-}} = \frac{1}{2}\left(\sqrt{\frac{u_-}{u_+}} - \sqrt{\frac{u_+}{u_-}}\right) \tag{7}$$

This is clearly a maximum since $\tan\delta \to 0$ as $v_- \to 0$ or $v_- \to \infty$.

From question 4 we have

$$\frac{u_+}{u_-} = \frac{\rho_-}{\rho_+} = \frac{\gamma-1}{\gamma+1} + \frac{2}{(\gamma+1)M_-^2} > \frac{\gamma-1}{\gamma+1}. \tag{8}$$

Moreover we deduced that in order for the entropy not to decrease on going through the shock (as required by the 2nd law of thermodynamics), we must have $M_-^2 \geq 1$. Thus, inspection of the above expression shows that the largest that u_+/u_- could be is 1, when $M_- = 1$.

The expression

$$\sqrt{\frac{u_-}{u_+}} - \sqrt{\frac{u_+}{u_-}} \tag{9}$$

is maximised when u_+/u_- is as small as possible, i.e. $u_+/u_- = (\gamma-1)/(\gamma+1)$, so we have

$$\tan\delta < \frac{1}{2}\left(\sqrt{\frac{\gamma+1}{\gamma-1}} - \sqrt{\frac{\gamma-1}{\gamma+1}}\right) = \frac{1}{\sqrt{\gamma^2-1}}. \tag{10}$$