B4.2 Functional Analysis II: Q2(b) – 2016

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(ii) Given:

- Z: normed space
- $\mathcal{E} \subset Z$ such that

$$\{\, \varphi(z) \,:\, z \in \mathcal{E} \,\} \text{ is bounded for each } \varphi \in Z^*.$$

Want: $\mathcal{E}$ is norm-bounded.

(iii) Given:

- X: a vector space.
- $\|\cdot\|_1$ and $\|\cdot\|_2$: two inequivalent norms on X.

Want: a linear functional on X which is continuous with respect to $\|\cdot\|_1$ and discontinuous with respect to $\|\cdot\|_2$.

Part (ii): Application of PUB

- $Y = Z^*$ is a Banach space.
- View $\mathcal{E}$ as a subset of $Y^* = Z^{**}$: Identify $x \in \mathcal{E}$ with $T_x \in Y^*$ defined by
$$T_x(\ell) = \ell(x) \text{ for } \ell \in Y.$$
Note that $\|x\| = \|T_x\|_{Y^*}$.
- By hypothesis, for each $\varphi \in Y$, the set $\{T(\varphi) : T \in \mathcal{E}\}$ is bounded. So by PUB, $\mathcal{E}$ is bounded as a subset of Y^*.
- This means there is some C such that $\|T\|_{Y^*} \leq C$ for all $T \in \mathcal{E}$. As $\|x\| = \|T_x\|_{Y^*}$, this gives the boundedness of $\mathcal{E}$ in Z.

Part (iii): Application of Part (ii)

- Observation: Let $B = \{x \in X : \|x\|_2 \leq 1\}$. Then B is unbounded with respect to $\|\cdot\|_1$.
- By (ii), there exists $\varphi \in (X, \|\cdot\|_1)^*$ such that $\{\varphi(x) : x \in B\}$ is unbounded.
- The first part means φ is continuous with respect to $\|\cdot\|_1$. The second part means φ is discontinuous with respect to $\|\cdot\|_2$.