B4.2 Functional Analysis II: Q1(a)(ii) – 2017

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Given:

- X: A Hilbert space.
- $K_1 \supset K_2 \supset K_3 \ldots$: nested sequence of non-empty closed convex subsets of X.

Want: Prove that

$$\bigcap_{m=1}^{\infty} K_m \neq \emptyset.$$

Approach 1 – Starting point: Closest point theorem + Parallelogram law

- For each m, there exists $x_m \in K_m$ which is closest to the origin.
- As K_1 is bounded, (x_m) is bounded.
- As $K_{m+1} \subset K_m$, we have $\|x_m\| \leq \|x_{m+1}\|$. So $(\|x_m\|)$ is a bounded monotone sequence, hence converges to some number, say d.

Approach 1 – Concluding point: Parallelogram law

- Claim: (x_m) is Cauchy. For this, we estimate $\|x_m - x_n\|$, say for $m > n$. We have

$$\|x_m - x_n\|^2 = \underbrace{2\|x_n\|^2}_{\to 2d^2} + \underbrace{2\|x_m\|^2}_{\to 2d^2} - \underbrace{4\|\underbrace{\frac{1}{2}(x_n + x_m)}_{\in K_n}\|^2}_{\geq 4\|x_n\|^2 \to 4d^2}.$$

  So $\limsup \|x_m - x_n\|^2 \leq 0$, i.e. (x_m) is Cauchy!
- Let $x = \lim x_m$. For each n, the sequence (x_m) eventually belongs to K_n, which is closed. So $x \in K_n$ for each n. So $x \in \cap K_m$.

Approach 2 – Weak convergence

- Take $x_m \in K_m$ arbitrarily. Then $(x_m) \subset K_1$ and hence is a bounded sequence.
- Weak sequential compactness then implies that a subsequence (x_{m_j}) converges weakly to some x.
- For each n, (x_{m_j}) eventually belongs to K_n which is closed and convex. By Mazur's theorem, x, being the weak limit of a sequence in this closed convex set, must be in K_n and we are done.
- Remark: If you are following this approach, you will need to prove (1) the weak sequential compactness property and (2) Mazur's theorem; as the problem does not tell you to freely use what you know.