B4.2 Functional Analysis II: Q2 – 2012

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Q2 – 2012

Given:

- $X = \left\{ f \in C([0,1];\mathbb{R}) \middle| f(0) = 0, f' \in L^2(0,1), f(t) = \int_0^t f'(s)\,\mathrm{d}s \ \forall t \in [0,1] \right\}$ is Hilbert with inner product

$$\langle f, g \rangle = \int_0^1 \left(f(t)g(t) + f'(t)g'(t) \right) \mathrm{d}t.$$

- $Y = C^2([0,1];\mathbb{R})$ is a dense subspace of X.
- For $t \in [0,1]$, $\phi_t \colon X \to \mathbb{R}$ is defined by $\phi_t(f) = f(t)$.

Want:

1. $\|\phi_t\|_* \leq t^{1/2}$.
2. Find $g \in Y$ such that $f(1) = \langle f, g \rangle$ for all $f \in X$.
3. Find the value of $\|\phi_1\|_*$.
4. Does there exist $h \in Y$ such that $f(\frac{1}{2}) = \langle f, h \rangle_X$ for all $f \in X$?

Part (i)

- ϕ_t is well-defined and linear on X.
- We compute, using Cauchy-Schwarz's inequality:

$$
\begin{aligned}
|\phi_t(f)| = |f(t)| &= \Big| \int_0^t f'(s)\,ds \Big| \\
&\leq \Big(\int_0^t 1^2\,ds \Big)^{1/2} \Big(\int_0^t |f'(s)|^2\,ds \Big)^{1/2} \leq t^{1/2} \|f\|.
\end{aligned}
$$

- So ϕ_t is bounded and $\|\phi_t\|_* \leq t^{1/2}$.

Part (ii): Integration by parts

- $\phi_1 \in X^*$, so by the Riesz representation theorem there exists a unique $g \in X$ such that $\phi_1(f) = \langle f, g\rangle$ for all $f \in X$, which means
$$f(1) = \int_0^1 [f(t)g(t) + f'(t)g'(t)]\,dt.$$
- We were told that $g \in Y$, so we can integrate by parts:
$$\begin{aligned} f(1) &= [fg']_0^1 + \int_0^1 f(t)[g(t) - g''(t)]\,dt \\ &= f(1)g'(1) + \int_0^1 f(t)[g(t) - g''(t)]\,dt. \end{aligned}$$

By inspection, we want $g - g'' = 0$, $g'(1) = 1$, plus $g(0) = 0$. So $g = \frac{\sinh x}{\cosh 1}$.

Part (iii): RRT

We have by the Riesz representation theorem that $\|\phi_1\|_* = \|g\|$. So

$$\|\phi_1\|_*^2 = \langle g, g \rangle = \phi_1(g) = \tanh 1.$$

So $\|\phi_1\|_* = (\tanh 1)^{1/2}$.

Part (iv): Testing and Orthogonality – 1

- As before, we are led to

$$f(1/2) = \int_0^1 [f(t)g(t) + f'(t)h'(t)]\,dt$$
$$= f(1)h'(1) + \int_0^1 f(t)[h(t) - h''(t)]\,dt.$$

- Taking $f \in C_c^\infty(0, 1/2)$ we have

$$0 = \int_0^{1/2} f(t)[h(t) - h''(t)]\,dt \text{ for all } f \in C_c^\infty(0, 1/2).$$

So $(h - h'')|_{(0,1/2)}$ is orthogonal to $C_c^\infty(0, 1/2)$ in $L^2(0, 1/2)$ and so is trivial.

Part (iv): Testing and Orthogonality – 2

- Applying the same argument to $(1/2, 1)$ we get $h - h'' \equiv 0$ in $(1/2, 1)$ and so in $(0, 1)$.
- So we obtain

$$f(1/2) = f(1)h'(1) \text{ for all } f \in X.$$

- This is impossible as for any given real numbers a and b, we can choose $f \in X$ with $f(1/2) = a$ and $f(1) = b$. So there is no such h.
- Remark: However by the Riesz representation theorem, there is some such $h \in X \setminus Y$.