B4.2 Functional Analysis II: Q2(b)(iv) – 2012

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Q2(b)(ii) – 2012

Given:

- X: A Hilbert space.
- $S \in \mathscr{B}(X)$, $\|S\| = 1$.

Want: Prove that

$$Ker(I - S^*) = Ker(I - S).$$

Proof of $Ker(I - S^*) \subset Ker(I - S)$

- Take $x \in Ker(I - S^*)$. Then $x = S^*x$.
- Now $\langle Sx, x\rangle = \langle x, S^*x\rangle = \|x\|^2 \geq \|x\|\|Sx\|$, as $\|S\| = 1$.
- By Cauchy-Schwarz' inequality, $|\langle Sx, x\rangle| \leq \|x\|\|Sx\|$.
- By the equality case in Cauchy-Schwarz' inequality, we then have $x = Sx$. So $x \in Ker(I - S)$.