B4.2 Functional Analysis II - Sheet 4 of 4

Read Chapter 4 & 5 and prove the few statements whose proofs were left out as an exercise. (Not to be handed in.)

Do:

Q0. (Not to be handed in.) This problem recalls a result in Part A Integration.

Let $f \in L^1_{loc}(\mathbb{R})$, $a \in \mathbb{R}$ and define

$$F(x) = \int_a^x f(t)\,dt.$$

(i) Show that F is continuous.

(ii) Show that if φ is a smooth function, then integration by parts hold:

$$\int_b^c F(x)\,\varphi'(x)\,dx = [F\varphi]_b^c - \int_b^c f(x)\,\varphi(x)\,dx.$$

[*Hint: First prove the statement for the case $b = a$ by expressing the left hand side as a repeated integral and then appealing to Fubini's theorem.*]

(iii) Show that if F is constant, then $f = 0$ a.e.

Q1. (a) Use Theorem 4.6.1 to prove the localisation property of Fourier series: if two (continuous) 2π-periodic functions f and g are equal in an open interval containing 0, then their Fourier series either both converge at 0 or both diverge at 0.

(b) In the lecture, we prove that there is a continuous function whose Fourier series diverges at 0. Use (a) to construct a continuous function whose Fourier series diverges at 0 and $\pi/2$.

Q2. Consider the system $\{e_n = \frac{1}{\sqrt{2\pi}} e^{inx}\}_{n\in\mathbb{Z}}$ as a subset of $X = L^1(-\pi, \pi)$.

(a) Show that $\|e_n\| = \sqrt{2\pi}$ for all n and $\|e_n - e_m\| = \frac{8}{\sqrt{2\pi}}$ for all $n \neq m$.

(b*) Show that $\{e_n = \frac{1}{\sqrt{2\pi}} e^{inx}\}_{n\in\mathbb{Z}}$ is a basis of $L^1(-\pi, \pi)$, i.e. the closed linear span of $\{e_n = \frac{1}{\sqrt{2\pi}} e^{inx}\}_{n\in\mathbb{Z}}$ is $L^1(-\pi, \pi)$.

Q3. Let X be the closed subspace of $C[-\pi,\pi]$ consisting of all continuous (on $[-\pi,\pi]$) functions f such that $f(-\pi) = f(\pi)$. For $n \in \mathbb{Z}$, define $e_n \in X$ by $e_n(t) = \frac{1}{\sqrt{2\pi}} e^{int}$ and let

$$\widehat{f}(n) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) e^{int} dt$$

for $f \in X$. Let $\{\alpha_n\}_{n\in\mathbb{Z}}$ be a sequence in $\mathbb{C}$, and assume that for each $f \in X$ there exists a unique element $g \in X$ such that $\widehat{g}(n) = \alpha_n \widehat{f}(n)$ for all $n \in \mathbb{Z}$. Let $Tf = g$.

(a) Show that T is linear and has closed graph. Deduce that $T \in \mathscr{B}(X)$.

(b) Show that $Te_n = \alpha_n e_n$ for all $n \in \mathbb{Z}$ and that the sequence $\{\alpha_n\}_{n\in\mathbb{Z}}$ is bounded.

(c) Show that there exists a bounded linear functional φ on X such that $\varphi(e_n) = \alpha_n$ for all $n \in \mathbb{Z}$.

Q4. Consider the right shift operator on sequences $R(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$. Show that as an operator on ℓ^2, R satisfies $\sigma_p(R) = \emptyset$, $\sigma_r(R) = \{\lambda : |\lambda| < 1\}$ and $\sigma_c(R) = \{\lambda : |\lambda = 1\}$.

[To put thing in perspective, compare Question 7 of Sheet 4 of B4.1 from MT: If we consider T as an operator on ℓ^∞, then $\sigma_p(R) = \emptyset$, $\sigma_r(R) = \{\lambda : |\lambda| \le 1\}$ and $\sigma_c(R) = \emptyset$.]

Q5. Let X be a complex Hilbert space and $A \in \mathscr{B}(X)$ be normal (i.e. $A^*A = AA^*$).

(a) Show that

$$\mathrm{rad}(\sigma(A)) = \|A\|.$$

Deduce that if P is a polynomial, then

$$\|P(A)\| = \sup_{\lambda\in\sigma(A)} |P(\lambda)|.$$

(b) Let P be a Laurent polynomial, i.e. $P(z) = \sum_k a_k\, z^k$ where the summation range is finite but may contains positive as well as negative powers. Show that if A is unitary, then

$$\|P(A)\| = \sup_{\lambda\in\sigma(A)} |P(\lambda)|.$$

Q6. Let X be a complex Hilbert space and S and T be two self-adjoint bounded linear operators on X.

(a) Let $\lambda \notin \sigma(T)$. Use the fact that $\sigma((T-\lambda I)^{-1}) = (\sigma(T)-\lambda)^{-1}$ (a form of spectral mapping theorem) and Gelfand's formula to show that

$$\|(T-\lambda I)^{-1}\| = \frac{1}{\operatorname{dist}(\lambda, \sigma(T))}.$$

Deduce that $I + (T-\lambda I)^{-1}(S-T)$ is invertible if

$$\|S-T\| < \operatorname{dist}(\lambda, \sigma(T)).$$

Hence, show under this latter assumption that $\lambda \notin \sigma(S)$.

(b) Use (a) to show that

$$\|S-T\| \geq \operatorname{dist}_H(\sigma(S), \sigma(T))$$

where the Hausdorff distance $\operatorname{dist}_H(A,B)$ between two closed subsets A and B of $\mathbb{C}$ is defined by

$$\operatorname{dist}_H(A,B) = \max(\sup_{a\in A}\min_{b\in B}|a-b|, \sup_{b\in B}\min_{a\in A}|a-b|).$$