B4.2 FAII – Sheet 4 of 4 – Solution guide

Q0. (Not to be handed in.) This problem recalls a result in Part A Integration.

Let $f \in L^1_{loc}(\mathbb{R})$, $a \in \mathbb{R}$ and define

$$F(x) = \int_a^x f(t)\,dt.$$

(i) Show that F is continuous.

"Solution". We need to show that if $x_n \to x$, then $F(x_n) \to F(x)$. Consider for example the case $x > a$. Write

$$F(x_n) = \int f(t)\chi_{[a,x_n]}(t)\,dt$$

and apply Lebesgue's dominated convergence theorem. □

(i) Show that if φ is a smooth function, then integration by parts hold:

$$\int_b^c F(x)\,\varphi'(x)\,dx = [F\varphi]_b^c - \int_b^c f(x)\,\varphi(x)\,dx.$$

[*Hint: First prove the statement for the case $b = a$ by expressing the left hand side as a repeated integral and then appealing to Fubini's theorem.*]

"Solution". When $b = a$, we compute, using Fubini's theorem,

$$\begin{aligned}\int_a^c F(x)\,\varphi'(x)\,dx &= \int_a^c \int_a^x f(t)\,\varphi'(x)\,dt\,dx = \int_a^c \int_t^c f(t)\,\varphi'(x)\,dx\,dt\\ &= \int_a^c f(t)\,(\varphi(c) - \varphi(t))\,dt\\ &= \varphi(c)\int_a^c f(t)\,dt - \int_a^c f(t)\,\varphi(t))\,dt\\ &= F(c)\varphi(c) - \int_a^c f(x)\,\varphi(x)\,dx.\end{aligned}$$

The conclusion is readily seen as $F(a) = 0$. □

(i) Show that if F is constant, then $f = 0$ a.e.

"Solution". When F is constant, we have $0 = F(b) - F(c) = \int_b^c f(t)\,dt = \int f(t)\,\chi_{[b,c]}(t)\,dt$ for all constants b and c. This implies that

$$\int f(t)\,\psi(t)\,dt = 0$$

for all simple functions ψ with compact support. In particular, for $\psi = \chi_{[-n,n]\cap\{f>0\}}$, we see that the set $[-n,n]\cap\{f>0\}$ has measure zero for all $n>0$ and so $\{f>0\}$ has measure zero. Likewise, $\{f<0\}$ has measure zero. □

Q1. (a) Use Theorem 4.6.1 to prove the localisation property of Fourier series: if two (continuous) 2π-periodic functions f and g are equal in an open interval containing 0, then their Fourier series either both converge at 0 or both diverge at 0.

"Solution". Apply Theorem 4.6.1 to the function $h = f - g$, which vanishes in a neighborhood around 0 and hence is Holder continuous in that interval. □

(b) In the lecture, we prove that there is a continuous function whose Fourier series diverges at 0. Use (a) to construct a continuous 2π-periodic function whose Fourier series diverges two given points $a \neq b$.

"Solution". Let f be a continuous 2π-periodic function such that $S_N(f)$ diverges at 0. Let g be a continuous 2π-periodic function such that $g(x) = f(x-a)$ in a neighborhood of a and equals and $g(x) = f(x-b)$ in a neighborhood of b. By (a), g satisfies the desired property. □

Q2. Consider the system $\{e_n = \frac{1}{\sqrt{2\pi}}e^{inx}\}_{n\in\mathbb{Z}}$ as a subset of $X = L^1(-\pi,\pi)$.

(a) Show that $\|e_n\| = \sqrt{2\pi}$ for all n and $\|e_n - e_m\| = \frac{8}{\sqrt{2\pi}}$ for all $n \neq m$.

"Solution". The first one is straightforward. For the second, we calculate

$$\begin{aligned}\int_{-\pi}^{\pi} |1 - e^{ikx}|\,dx &= \int_{-\pi}^{\pi} \sqrt{2}(1-\cos kx)^{1/2}\,dx \\ &= \int_{-\pi}^{\pi} 2\left|\sin\frac{kx}{2}\right|dx = 2k\int_0^{2\pi/k} \sin\frac{kx}{2} = 8,\end{aligned}$$

from which the conclusion follows. □

(b*) Show that $\{e_n = \frac{1}{\sqrt{2\pi}}e^{inx}\}_{n\in\mathbb{Z}}$ is a basis of $L^1(-\pi,\pi)$, i.e. the closed linear span of $\{e_n = \frac{1}{\sqrt{2\pi}}e^{inx}\}_{n\in\mathbb{Z}}$ is $L^1(-\pi,\pi)$.

"Solution". We know that $\{e_n = \frac{1}{\sqrt{2\pi}} e^{inx}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(-\pi,\pi)$: Namely, for any $f \in L^2(-\pi,\pi)$, its Fourier series $\mathscr{F}(f) = \sum_n c_n e^{inx}$ converges in $L^2(-\pi,\pi)$.

Let $f \in L^1(-\pi,\pi)$ and fix some $\varepsilon > 0$. We need to show that there is a finite sum $S_N = \sum_{|n|\le N} d_n\, e^{inx}$ such that

$$\|f - S_N\|_{L^1(-\pi,\pi)} \le \varepsilon.$$

First select $g \in C[-\pi,\pi]$ such that $\|f-g\|_{L^1(-\pi,\pi)} \le \varepsilon/2$. Since $g \in L^2(-\pi,\pi)$, we can then select $S_N = \sum_{|n|\le N} d_n\, e^{inx}$ such that

$$\|g - S_N\|_{L^2(-\pi,\pi)} \le \frac{\varepsilon}{2\sqrt{2\pi}}.$$

By Cauchy-Schwarz inequality, we have

$$\|g - S_N\|_{L^1(-\pi,\pi)} \le \|1\|_{L^2(-\pi,\pi)}\|g - S_N\|_{L^2(-\pi,\pi)} \le \varepsilon/2.$$

The conclusion follows from triangle inequality. □

Q3. Let X be the closed subspace of $C[-\pi,\pi]$ consisting of all continuous (on $[-\pi,\pi]$) functions f such that $f(-\pi) = f(\pi)$. For $n \in \mathbb{Z}$, define $e_n \in X$ by $e_n(t) = \frac{1}{\sqrt{2\pi}} e^{int}$ and let

$$\hat{f}(n) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) e^{-int} dt$$

for $f \in X$. Let $\{\alpha_n\}_{n\in\mathbb{Z}}$ be a sequence in $\mathbb{C}$, and assume that for each $f \in X$ there exists a unique element $g \in X$ such that $\hat{g}(n) = \alpha_n \hat{f}(n)$ for all $n \in \mathbb{Z}$. Let $Tf = g$.

(a) Show that T is linear and has closed graph (and so is a bounded operator on X).

"Solution". Assume that $f_k \to f$ and $Tf_k = g_k \to g$. Then

$$\hat{g}(n) = \lim_{k\to\infty} \hat{g}_k(n) = \lim_{k\to\infty} \alpha_n \hat{f}_k(n) = \alpha_n \hat{f}(n) \text{ for every } n.$$

So $g = Tf$. □

(b) Show that $Te_n = \alpha_n e_n$ for all $n \in \mathbb{Z}$ and that the sequence $\{\alpha_n\}_{n\in\mathbb{Z}}$ is bounded.

"Solution". That $Te_n = \alpha_n e_n$ is clear. Now $\|e_n\| = \frac{1}{\sqrt{2\pi}}$ and $\|Te_n\| = \frac{\alpha_n}{\sqrt{2\pi}}$. Hence

$$|\alpha_n| = \sqrt{2\pi}\|Te_n\| \leq \sqrt{2\pi}\|T\|\|e_n\| = \|T\|,$$

which gives the second assertion. □

(c) Show that there exists a bounded linear functional φ on X such that $\varphi(e_n) = \alpha_n$ for all $n \in \mathbb{Z}$.

"Solution". Let $\varphi(f) = \sqrt{2\pi}\, Tf(0)$. Then $\varphi(e_n) = Te_n(0) = \sqrt{2\pi}\alpha_n e_n(0) = \alpha_n$, as desired.

[It is tempted to say that, as $f = \sum \hat{f}(n)\, e_n$, we can let

$$\varphi(f) = \sum \alpha_n \hat{f}(n)$$

The problem is that the convergence of $\sum \hat{f}(n)\, e_n$ to f is not uniform, so this is not allowed.] □

Q4. Consider the right shift operator on sequences $R(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$. Show that as an operator on ℓ^2, R satisfies $\sigma_p(R) = \emptyset$, $\sigma_r(R) = \{\lambda : |\lambda| < 1\}$ and $\sigma_c(R) = \{\lambda : |\lambda = 1\}$.

[To put thing in perspective, compare Question 7 of Sheet 4 of B4.1 from MT: If we consider T as an operator on ℓ^∞, then $\sigma_p(R) = \emptyset$, $\sigma_r(R) = \{\lambda : |\lambda| \leq 1\}$ and $\sigma_c(R) = \emptyset$.]

"Solution". It is easy to see that $\sigma_p(R) = \emptyset$.

The adjoint of R is the left shift L. The point spectrum of L is readily seen to be $\{\lambda : |\lambda| < 1\}$. The spectrum of L is a closed bounded subset of $\{\lambda : |\lambda| \leq \|L\| = 1\}$ and so $\sigma(L) = \{\lambda : |\lambda| \leq 1\}$.

We have $\sigma_r(L) \subset \sigma_p'(R) = \emptyset$. Hence $\sigma_r(L)$ is empty and so $\sigma_c(L) = \sigma(L) \setminus (\sigma_p(L) \cup \sigma_r(L)) = \{\lambda : |\lambda| = 1\}$.

Now as $\overline{\mathrm{Im}\,(\lambda I - R)}^{\perp} = \mathrm{Ker}\,(\bar{\lambda} I - L)$ which is non-trivial if $|\lambda| < 1$ and trivial if $|\lambda| = 1$. It follows that $\mathrm{Im}\,(\lambda I - R)$ is non-dense if $|\lambda| < 1$ and dense if $|\lambda| = 1$, i.e. $\sigma_r(R) = \{\lambda : |\lambda| < 1\}$ and $\sigma_c(R) = \{\lambda : |\lambda| = 1\}$. □

Q5. Let X be a complex Hilbert space and $A \in \mathscr{B}(X)$ be normal (i.e. $A^*A = AA^*$).

(a) Show that

$$\mathrm{rad}(\sigma(A)) = \|A\|.$$

Deduce that if P is a polynomial, then

$$\|P(A)\| = \sup_{\lambda\in\sigma(A)} |P(\lambda)|.$$

"Solution". As $T := A^*A$ is self-adjoint, $\|T^n\| = \|T\|^n$. Hence, by Gelfand's formula

$$\mathrm{rad}(\sigma(T)) = \|T\|.$$

An induction gives $\|T^n\| = \|A^n\|^2$. By Gelfand's formula, this implies

$$\mathrm{rad}(\sigma(T)) = \mathrm{rad}(\sigma(A))^2.$$

Combining the two identities and the fact that $\|T\| = \|A\|^2$, we get the first conclusion.

Now, as A is normal, so is $P(A)$. Hence

$$\|P(A)\| = \mathrm{rad}(\sigma(P(A)) = \sup_{\lambda\in\sigma(P(A))} |\lambda|.$$

But $\sigma(P(A)) = P(\sigma(A))$ by the spectral mapping theorem, so the second conclusion follows. □

(b) Let P be a Laurent polynomial, i.e. $P(z) = \sum_k a_k\, z^k$ where the summation range is finite but may contains positive as well as negative powers. Show that if A is unitary, then

$$\|P(A)\| = \sup_{\lambda\in\sigma(A)} |P(\lambda)|.$$

"Solution". Write $P(z) = z^{-N}\, Q(z)$ where $N \geq 0$ and Q is a polynomial. Since A is unitary, $\|P(A)\| = \|Q(A)\|$. Hence by (a),

$$\|P(A)\| = \sup_{\lambda\in\sigma(A)} |Q(\lambda)|.$$

As $\sigma(A)$ is a subset of the unit circle, we have $|Q(\lambda)| = |P(\lambda)|$ on $\sigma(A)$ and so the conclusion follows. □

Q6. Let X be a complex Hilbert space and S and T be two self-adjoint bounded linear operators on X.

(a) Let $\lambda \notin \sigma(T)$. Use the fact that $\sigma((T-\lambda I)^{-1}) = (\sigma(T)-\lambda)^{-1}$ (a form of spectral mapping theorem) and Gelfand's formula to show that

$$\|(T-\lambda I)^{-1}\| = \frac{1}{\operatorname{dist}(\lambda, \sigma(T))}.$$

Deduce that $I + (T-\lambda I)^{-1}(S-T)$ is invertible if

$$\|S-T\| < \operatorname{dist}(\lambda, \sigma(T)).$$

Hence, show under this latter assumption that $\lambda \notin \sigma(S)$.

"Solution". We have that

$$\begin{aligned}
\|(T-\lambda I)^{-1}\| &= \operatorname{rad}(\sigma((T-\lambda I)^{-1})) = \sup_{\zeta \in \sigma((T-\lambda I)^{-1})} |\zeta| \\
&= \sup_{\zeta \in (\sigma(T)-\lambda)^{-1}} |\zeta| = \Big(\inf_{\zeta \in \sigma(T)-\lambda} |\zeta| \Big)^{-1} \\
&= \frac{1}{\operatorname{dist}(\lambda, \sigma(T))}.
\end{aligned}$$

Hence, if $\|S-T\| < \operatorname{dist}(\lambda, \sigma(T))$, then $K := (T-\lambda I)^{-1}(S-T)$ satisfies $\|K\| < 1$ and so $I+K$ is invertible. This implies that $(T-\lambda I)(I+K) = S-\lambda I$ is invertible and so $\lambda \notin \sigma(S)$. □

(b) Use (a) to show that

$$\|S-T\| \geq \operatorname{dist}_H(\sigma(S), \sigma(T))$$

where the Hausdorff distance $\operatorname{dist}_H(A, B)$ between two closed subsets A and B of $\mathbb{C}$ is defined by

$$\operatorname{dist}_H(A, B) = \max(\sup_{a \in A} \min_{b \in B} |a-b|, \sup_{b \in B} \min_{a \in A} |a-b|).$$

"Solution". Suppose by contradiction that the conclusion fails. We may assume without loss of generality that

$$\|S-T\| < \sup_{a \in \sigma(S)} \min_{b \in \sigma(T)} |a-b| = \sup_{a \in \sigma(S)} \operatorname{dist}(a, \sigma(T)).$$

Then, we can select $\lambda \in \sigma(S)$ such that

$$\|S-T\| < \operatorname{dist}(\lambda, \sigma(T)).$$

(This implies that $\operatorname{dist}(\lambda, \sigma(T)) > 0$ and so $\lambda \notin \sigma(T)$.) By (a), this implies that $\lambda \notin \sigma(S)$, which is a contradiction. □