B4.2

With acknowledgement to Professor H A Priestly,
whose the last year Lecture Notes on Hilbert Space
are reprinted here with minor changes

March 6, 2017

1 Hilbert Spaces

1.1 Inner Product Spaces

Definition 1.1. *A linear vector space X over scalar field $\mathbb{C}$ (or $\mathbb{R}$) is called Inner Product Space (ISP) if there exists a function $< \cdot, \cdot >: X \times X \to \mathbb{C}$ (or $\mathbb{R}$) having the following properties:*
(i) $< x, y >= \overline{< y, x >}$ *(or* $< x, y >=< y, x >$*)* $\forall x, y \in X$
(ii) $< \lambda x, y >= \lambda < x, y >$ $\forall \lambda \in \mathbb{C}$ *(or* $\mathbb{R}$*)* $\forall x, y \in X$
(iii) $< x + y, z >=< x, z > + < y, z >$ $\forall x, y, z \in X$
(iv) $< x, x > \in \mathbb{R}_+ := \{t \in \mathbb{R} : \ t \geq 0\}, \ < x, x >= 0 \Leftrightarrow x = 0.$

Function $< \cdot, \cdot >$ is called inner (scalar) product. Properties (i)-(iv) are called axioms of inner product.

The inner product generates a norm $\|x\| = \sqrt{< x, x >}$ called the associated norm. So, any IPS is a normed space and thus a metric space with the distance $d(x, y) = \sqrt{< x - y, x - y >}$.

ISP is called a Hilbert space if it is a Banach space with respect to the associated norm.

1.2 Basic Facts about Inner Product

Let X be a ISP with the inner product $< \cdot, \cdot >$ and the associated norm $\| \cdot \|$.
(i) **Cauchy-Schwarz inequality**

$$| < x, y > | \leq \|x\| \|y\|$$

with the equality if and only if x and y are linearly dependent.
(ii) $< \cdot, \cdot >$ is continuous function from $X \times X$ to the corresponding scalar field.
(iii) **Parallelogram Law**

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \forall x, y \in X.$$

In fact, Parallelogram Law is a sufficient and necessary condition for a norm $\|x\|$ to be the associated norm for an inner product on X.
(iv) **Polarisation** Retrieving the inner product from the norm satisfying Parallelogram Law:

$$< x, y >= \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$$

for the real scalar field and

$$< x, y >= \frac{1}{4}(\|x+y\|^2 + i\|x+iy\|^2 - \|x-y\|^2 - i\|x-iy\|^2)$$

for the complex scalar field.

1.3 Examples

1. Eucledian Space $\mathbb{C}^n$ $(\mathbb{R}^n)$ with the inner product

$$< x, y >= \sum_{i=1}^{n} x_i \overline{y}_i, \quad x = (x_1, x_2, ..., x_n) = (x_i), y = (y_i) \in \mathbb{C}^n(\mathbb{R}^n).$$

It is a Hilbert space as any finite dimensional normed space is a Banach space.
2. Space $l_2 := \{x = (x_1, x_2, ..., x_n, ...) = (x_i) : \sum\limits_{n=1}^{\infty} |x_n|^2 < \infty\}$. It is a Hilbert space with the inner product

$$< x, y >= \sum_{n=1}^{\infty} x_n \overline{y}_n.$$

3. Lebesgue's space $L_2(E)$ consisting of all measurable functions $f : E \to \mathbb{C} \cup \{\infty\}$ such that $\int\limits_E |f|^2 dx < \infty$, where E is a measurable subset in $\mathbb{R}^n$. It is an ISP for the inner product

$$< f, g >:= \int\limits_E f\overline{g} dx.$$

The completeness of $L_2(E)$ with respect to the associated norm is a particular case of the Riesz-Fiesher theorem about completeness of Lebesgue's Spaces $L_p(E)$, see Integration course.

In fact we have much more general statement. Let (E, Σ, μ) be an arbitrary measure space. Then $L_2(E, \Sigma, \mu)$ is a Hilbert space. The important example is the Lebesgue-Stieltjes measure given by $\mu(A) = \int\limits_A p dx$, for any measurable subset $A \subseteq E$, with a non-negative weight function p. The corresponding set is denoted $L_2(E, pdx)$. It is a Hilbert space with the inner product

$$< f, g >= \int\limits_E f\overline{g} d\mu = \int\limits_E f\overline{g} p dx.$$

4.

Proposition 1.2. *A closed subset of a Hilbert space is a Hilbert space.*

Proof. It follows from the fact the closed subspace of a complete metric space is a complete metric space. □

Let $|E| < \infty$ and

$$\overline{L}_2(E) := \{f \in L_2(E) : \frac{1}{|E|} \int\limits_E f dx = 0\}.$$

Indeed, $\overline{L}_2(E)$ is closed in $L_2(E)$. So, $\overline{L}_2(E)$ is a proper closed subset in $L_2(E)$ therefore it is a Hilbert space itself with the inner product

$$< f, g >= \int\limits_E f\overline{g} dx.$$

5. Bergman space: let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$ equipped with area measure. The subspace $A^2(\mathbb{D})$ of $L_2(\mathbb{D})$ consisting of those holomorphic (in $\mathbb{D}$) functions which are in $L_2(\mathbb{D})$ is closed so it is a Hilbert space for the L_2-inner product, see Qn 3 in Problem Sheets.

6. The Hardy space: Let Y, sometimes denoted by $H^2(\mathbb{T})$, be the space of all those $f \in L_2(-\pi, \pi)$ with the Fourier series of the form $\sum\limits_{n=0}^{\infty} a_n e^{int}$ (so $a_n = 0$ for all $n < 0$). Then Y is a closed subspace of $L_2(-\pi, \pi)$. This example appears in different, but equivalent, guise later.

7. Sobolev space $H^1(a,b)$. We say that $u \in H^1(a,b)$ if and only if $u \in L_2(a,b)$ and there exists a function $v \in L_2(a,b)$ such that

$$u(x) = A + \int_a^x v(y)dy \tag{1.1}$$

for a.a. $x \in]a,b[$ and for some constant A.

Just from the definition and properties of Lebesgue's integral (explain which properties should be used), it follows that $u \in C([a,b]$ and

$$u(x) - u(y) = \int_y^x v(s)ds \quad \forall x, y \in [a,b].$$

It is also easy to see that, given $u \in H^1(a,b)$, there exists the only one function v, satisfying identity (1.1). Indeed, if not, then we have

$$\int_y^x (v_1(s) - v_2(s))ds = 0 \quad \forall x, y \in [a,b].$$

Since, for a.a. $\in]a,b[$,

$$\lim_{\delta \to 0} \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} (v_1(s) - v_2(s))ds = v_1(x) - v_2(x).$$

Thus v_1 and v_2 belong to the same equivalence class.

If v is a continuous function, then

$$u'(x) = \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} = \lim_{h \to 0} \int_h^{x+h} v(y)dy = v(x).$$

So, $H^1(a,b)$ contains $C^1([a,b])$. Moreover, the above limit exists a.e. in $]a,b[$ even if $v \in L_2(a,b)$. In other words, functions $u \in H^1(a,b)$ have the usual derivative a.e. in $]a,b[$. It makes sense to call v "weak" or "generalised" derivative of u in $]a,b[$ and use the classical notation letting $v = u'$.

$H^1(a,b)$ is a real Hilbert space with respect to the following inner product

$$< u, v >= \int_a^b (uv + u'v')dx.$$

In order to show that one needs to prove that $< \cdot, \cdot >$ satisfies all axioms of IPS (which is obvious) and that $H^1(a,b)$ is complete with respect to the associated norm, see Qn 4 in Problem Sheets.

1.4 Orthogonality

Let be X be an IPS with the inner product $< \cdot, \cdot >$. x is orthogonal to y if $< x, y > 0$. Obviously, if x is orthogonal to y, then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ which is a version of the Pythagoras theorem.

Let $Y \subseteq X$. Then

$$Y^\perp := \{x \in X : < x, y >= 0 \ \forall y \in Y\}$$

is the orthogonal compliment of Y in X.

Proposition 1.3. *Let Y be a subset of IPS X. Then*
(i) $Y^\perp$ is a closed subspace of X.
(ii) $Y \subseteq Y^{\perp\perp}$.
(iii) If $Y \subseteq Z \subseteq X$, then $Z^\perp \subseteq Y^\perp$.
(iv) $(\overline{\operatorname{span} Y})^\perp = Y^\perp$.
(v) If Y and Z are subsets of X such that $X = Y + Z$ and $Z \subseteq Y^\perp$, then $Y^\perp = Z$.

Proof. Exercise. □

1.5 Closest Point Theorem

Theorem 1.4. *Let C be a non-empty closed convex subset set of a Hilbert space X. Let $x \in X$. There exists a unique point $y_0 \in Y$ such that $\|x - y\| \geq \|x - y_0\|$ for all $y \in C$.*

Proof. Let $\delta := \inf\{\|x - y\| : y \in C\} \geq 0$ and let $y_n \in$, $n = 1, 2, ...$, be a minimising sequence, i.e., $\|x - y_n\| \to \delta$ as $n \to \infty$. Let $z_n = x - y_n$. By convexity of C, $\frac{1}{2}(y_n + y_m) \in C$ and so

$$\|z_n + z_m\|^2 = 4\|x - \frac{1}{2}(y_m + y_n)\|^2 \geq 4\delta^2.$$

According to Parallelogram Law

$$\|z_n - z_m\|^2 = -\|z_m + z_n\|^2 + 2\|z_m\|^2 + 2\|z_n\|^2 \leq 2\|z_m\|^2 + 2\|z_n\|^2 - 4\delta^2 \to 0$$

as $m, n \to \infty$. Hence y_n is a Cauchy sequence and so converges. The limit y_0 belongs to C as C is closed, $\|x - y_0\| = \delta$ by continuity of the norm.

Uniqueness follows from Parallelogram Law in the same manner as above. □

1.6 Projection Theorem

Theorem 1.5. *Let Y be a closed subspace of a Hilbert space X. Then $X = Y \oplus Y^\perp$.*

Proof. Certainly, $Y \cap Y^\perp = \{0\}$. Let us show that $X = Y + Y^\perp$. Take any $x \in X$ and note that Y is a non-empty and convex as well. Choose $y_0 \in Y$ according to Theorem 1.4. We claim that $x - y_0 \in Y^\perp$. Indeed, for any $y \in Y$ and any $t \in \mathbb{R}$, we have $y_0 + ty \in Y$ and by the definition of y_0

$$\|y_0 - x\|^2 \leq \|y_0 - x + ty\|^2 = \|y_0 - x\|^2 + 2t\text{Re} < y_0 - x, y > + t^2\|y\|^2.$$

So, $0 \leq 2t\text{Re} < y_0 - x, y > + t^2\|y\|^2$ for all t. Dividing the latter inequality by positive t and then passing t to zero, we conclude that Re $< y_0 - x, y >= 0$ for all $y \in Y$. This completes the proof if X is a real Hilbert space.

If the scalar field is complex, we proceed as before with t replaced by it. □

Corollary 1.6. *Let Y be a subspace of a Hilbert space X*
(i) The following are equivalent:
(a) Y is closed in X,
(b) $X = Y + Y^\perp$,
(c) $Y = Y^{\perp\perp}$.
(ii) $Y^{\perp\perp} = \overline{Y}$.
(iii) The following are equivalent:
(a) Y is dense in X,
(b) $Y^\perp = \{0\}$.

Proof. Exercise. You will need the Projection Theorem and also the facts Proposition 1.3. □

1.7 Projecting onto a closed subspace

Let Y be a closed subspace of a Hilbert space X.

Let $P_Y x = y$, where $x = y + z$ is the unique decomposition of x, with $y \in Y$ and $z \in Y^{\perp}$. Then, by Pythagoras Theorem,

$$\|x\|^2 = \|P_Y x\|^2 + \|x - P_Y x\|^2 \geq \|P_Y x\|^2.$$

Hence the linear map $P_Y : X \to X$ is bounded, and of the norm at most one. So, by definition $P_Y x$ is the closest point for x to Y and the required minimum distance is simply $\|z\|$.

Here are some examples.

(a) Let $X = L_2(-\pi, \pi)$, and Y be the space of all $f \in X$ such that $f(t) = 0$ for a.a. $t \in]0, \pi[$. Then $Y^{\perp}$ is the space of all $f \in X$ such that $f(t) = 0$ for a.a. $t \in]-\pi, 0[$, and $P_Y f = f\chi_{]-\pi,0[}$.

(b) Let $X \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{G}$ be a sub-σ-algebra of $Y = L_2(\Omega, \mathcal{G}, \mathbb{P})$. For an $\mathcal{F}$-measurable random variable ξ, $P_Y \xi$ is the conditional expectation $\mathbb{E}(\xi|\mathcal{G})$ of ξ with respect to $\mathcal{G}$.

2 Linear functionals and linear operators in Hilbert spaces

2.1 Riesz Representation Theorem

Let X be a Hilbert space, X^* is dual of X, i.e., consists of all linear (bounded) functionals $x^* \in X^* : X \to \mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). X^* is a Banach space with respect to the norm $\|x^*\|_* = \sup\limits_{\|x\| \leq 1} |x^*(x)|$.

Obviously, $y \mapsto < y, x >$ is a linear functional on X. By the Cauchy-Schwarz inequality, $| < y, x > | \leq \|y\| \|x\|$ and thus it is bounded and belongs to X^*, i.e., $x^*(y) = < y, x >$ and $\|x^*\|_* = \|x\|$. Converse is also true.

Theorem 2.1. *Given $x^* \in X^*$, there exists a unique $x \in X$ such that $x^*(y) = < y, x >$ for all $y \in Y$ and $\|x^*\| = \|x\|$.*

Remark 2.2. *In the case of real Hilbert spaces, the above statement means that there exists an isometric isomorphism $\pi : X \to X^*$ such that $\pi x = x^*$ is equivalent to $x^*(y) = < y, x >$ for all $y \in X$ and $\|x^*\|_* = \|x\|$. So, spaces X and x^* are topologically equivalent, i.e., they are the same up isometric isomorphism. It is notated as $X^* \cong X$ or even just $X^* = X$.*

Proof. $x^* = 0$, then $x = 0$. Now, assume that $x^* \neq 0$. Let $Y = \operatorname{Ker} x^*$. It is a closed subspace of X. By Theorem 1.5, $X = Y \oplus Y^\perp$. Take non-zero $y^\perp \in V^\perp$. Then, for any $z \in X$, we have,

$$z - \frac{x^*(z)y^\perp}{x^*(y^\perp)} \in Y = \operatorname{Ker} x^*.$$

Thus

$$< z, y^\perp > - \frac{x^*(z)\|y^\perp\|^2}{x^*(y^\perp)} = 0 \quad \forall z \in X.$$

So,

$$x = y^\perp \frac{\overline{x^*(y^\perp)}}{\|y^\perp\|^2}.$$

Uniqueness is obvious.

The norm identity $\|x^*\|_* = \|x\|$ can be proven as follows. By the Cauchy-Schwarz inequality, $x^*(y) =< y, x >\leq \|y\|\|x\|$ which implies $\|x^*\|_* \leq \|x\|$. On the other hand, $x^*(x) = \|x\|^2 \leq \|x^*\|\|x\|$ and thus $\|x\| \leq \|x^*\|_*$. This completes the proof. □

2.2 Weak convergence in Hilbert spaces

Definition 2.3. *Let X be a Hilbert space and $x^{(n)} \in X$, $n = 1, 2, ...$, be a sequence and $x \in X$. $x^{(n)}$ converges to x weakly in X ($x^{(n)} \rightharpoonup x$) if $< x^{(n)}, y >\to< x, y >$ for any $y \in X$.*

Proposition 2.4. *The following are true:*
(i) $x^{(n)} \to x \Rightarrow x^{(n)} \rightharpoonup x$;
(ii) $x^{(n)} \rightharpoonup x$ and $\|x^{(n)}\| \to \|x\| \Rightarrow x^{(n)} \to x$;
(iii) $x^{(n)} \rightharpoonup x \Rightarrow x^{(n)}$ is bounded;
(iv) $x^{(n)} \rightharpoonup x \Rightarrow \liminf_{n\to\infty} \|x^{(n)}\| \geq \|x\|$.

Proof. (i) and (ii) is an easy exercise. (iii) will be proven later, see Banach-Steinhaus Theorem. In order to prove (iv), note that there exists a subsequence $x^{(n_k)}$ such that

$$A = \liminf_{n\to\infty} \|x^{(n)}\| = \lim_{k\to\infty} \|x^{(n_k)}\|.$$

Thus we have

$$A = \lim_{k\to\infty} \sup\{< x^{(n_k)}, y >: \ \|y\| = 1\} \geq \lim_{k\to\infty} < x^{(n_k)}, y >=< x, y >$$

for all $\|y\| = 1$. Taking supremum in RHS, we complete our proof of (iv). $\square$

Remark 2.5. *The statement inverse to (i) in general in not true. Indeed, let $X = l^2$. Consider the sequence $x^{(n)}$ with $x_i^{(n)} = 0$ if $i \neq n$ and $x_n^{(n)} = 1$. This sequence converges weakly to zero. Indeed, for any $\|y\| < \infty$, $< x^{(n)}, y >= y_n \to 0$ as $n \to \infty$. On the other hand, sequence does not converge since $\|x^{(n)} - x^{(m)}\| = \sqrt{2}$ if $\neq m$. For the same reason, this sequence does not contain a converging subsequence.*

The statement below is a version of Bolzano-Weierstrass Lemma in infinite dimensional Hilbert spaces. It is called the Sequential Weak Compactness Theorem.

Theorem 2.6. *Any bounded sequence of a Hilbert space contains a weakly converging subsequence.*

Proof. Let X be a Hilbert space and let $x^{(n)}$ be a bounded sequence of X, i.e.,

$$M := \sup_{n \geq 1} \|x^{(n)}\| < \infty.$$

The proof is based on the celebrated diagonal Cantor procedure. Let us describe it. Consider the sequence $\{< x^{(n)}, x^{(1)} >\}_{n \in \mathbb{N}}$. By the Cauchy-Schwarz inequality, we have $| < x^{(n)}, x^{(1)} > | \leq M\|x^{(1)}\|$ and thus the abovesequence is bounded. By Bolzano-Weiestrass Lemma, there exists a subsequence $\{n_j^1\}_{j \in \mathbb{N}}$ of natural numbers such that $\lim\limits_{j \to \infty} < x^{(n_j^1)}, x^{(1)} >$ exists. Then we consider another bounded scalar sequence $\{< x^{(n_j^1)}, x^{(2)} >\}_{j \in \mathbb{N}}$. By the above arguments, there exists a subsequence $\{n_j^2\}_{j \in \mathbb{N}}$ of the sequence $\{n_j^1\}_{j \in \mathbb{N}}$ such that $\lim\limits_{j \to \infty} < x^{(n_j^2)}, x^{(2)} >$ exists and, of course, $\lim\limits_{j \to \infty} < x^{(n_j^2)}, x^{(1)} >$ exists as well, since $\{x^{(n_j^2)}\}$ is a subsequence of $\{x^{(n_j^1)}\}$. Proceeding in the same way, we find an infinite sequence of nested subsequences $\{n_j^k\}_{j \in \mathbb{N}}$, $k = 1, 2, ...$, so that, for any k, $\{n_j^k\}_{j \in \mathbb{N}}$ is a subsequence of $\{n_j^{k-1}\}_{j \in \mathbb{N}}$ and

$$\lim_{j \to \infty} < x^{(n_j^k)}, x^{(m)} >$$

exists for all $m = 1, 2, .., k$. Let $y^{(k)} = x^{(n_k^k)}$. This subsequence of our original sequence and morever

$$\lim_{k \to \infty} < y^{(k)}, x^{(m)} >$$

exists for all m. Indeed, let us fix m arbitrarily, then $\{< y^{(k)}, x^{(m)} >\}_{k=m}^{\infty}$ is a subsequence of the converging sequence $\{< x^{(n_j^k)}, x^{(m)} >\}_{j=m}^{\infty}$.

So, we have, for any m,

$$< y^{(k)} - y^{(j)}, x^{(m)} >\to 0$$

as k and j tend to infinity. This certainly implies

$$< y^{(k)} - y^{(j)}, y >\to 0 \qquad \forall y \in Y := \text{span}(\{x^{(n)}\}_{n=1}^{\infty}) \tag{2.1}$$

as k and j tend to infinity. Let us show that (2.1) is valid for any $y \in \overline{Y}$. Indeed, for any $y \in \overline{Y}$ and for any $z \in Y$, we have

$$| < y^{(k)} - y^{(j)}, y > | \le | < y^{(k)} - y^{(j)}, z > |+$$

$$+| < y^{(k)} - y^{(j)}, y - z > | \le | < y^{(k)} - y^{(j)}, z > | + 2M\|y - z\|.$$

Now, we can make the second term of RHS to be small by choosing z and then the first term is going to be small for sufficiently big k and j, see (2.1).

By the Projection Theorem, $X = \overline{Y} \oplus (\overline{Y})^{\perp}$. So, for any $y \in X$, we have $y = z + z^{\perp}$ with $z \in \overline{Y}$ and $z^{\perp} \in (\overline{Y})^{\perp}$ and thus

$$< y^{(k)} - y^{(j)}, y >=< y^{(k)} - y^{(j)}, z >\to 0$$

as k and j tend to infinity. So, for any $y \in X$, there exists

$$\lim_{k\to\infty} < y^{(k)}, y >$$

and thus

$$\lim_{k\to\infty} \overline{< y^{(k)}, y >} = \lim_{k\to\infty} < y, y^{(k)} >=: l(y).$$

Obviously, l is a linear functional on X. It is bounded since $|l(y)| \le M\|y\|$. By Theorem 2.1, there exists $x \in X$ such that $l(y) =< y, x >$. So,

$$\lim_{k\to\infty} < y^{(k)}, y >= \overline{< y, x >} =< x, y > .$$

□

2.3 Adjoint operators in Hilbert spaces

Let X and Y be two Hilbert spaces and $\mathcal{B}(X, Y)$ be the Banach space of linear (bounded) operators defined on X with values in Y.

Consider $A \in \mathcal{B}(X, Y)$. Then $x \mapsto < Ax, y >_Y$ is a linear bounded functional on X. By the Riesz Representation Theorem, the exists a unique $l(y) \in X$ such that $< Ax, y >_Y = < x, l(y) >_X$. It is easy to see that $l : Y \to X$ is a linear operator which is denoted by A^* and called adjoint operator of A. So, we have by definition $< Ax, y >_Y = < x, A^*y >_X$. The following is easy to check.
(i) A^* is unique.
(ii) $A^* \in \mathcal{B}(Y, X)$.
(iii) $\|A^*\| = \|A\|$.
Indeed, (i) is obvious. For (ii), we let $x = A^*y$ and have $\|A^*y\|_X^2 \leq \|A\|\|A^*y\|_X\|y\|_Y$ and thus $\|A^*\| \leq \|A\|$. To get an opposite inequality, we let $y = Ax$, which gives (iii).
(iv) Let $T \in \mathcal{B}(X, Y)$ and $S \in \mathcal{B}(Y, Z)$. Then $(ST)^* = T^*S^*$ (Exercise).
(v) If $T \in \mathcal{B}(X, Y)$, then $T^{**} = T$.
Indeed, we have

$$< Tx, y >_Y = < x, T^*y >_X = \overline{< T^*y, x >_X} = \overline{< y, (T^*)^*x >_Y} = < T^{**}x, y >_Y$$

for all $x \in X$ and all $y \in Y$.

Consider several simple examples of ajoint operators.
(i) Let T be the following integral operator:

$$(Tf)(x) := \int_0^1 k(x, y) f(y) dy$$

in spaces $X = Y = L_2(0, 1; \mathbb{C})$, assuming for simplicity that the kernel $k :]0, 1[^2 \to]0, 1[$ is bounded measurable function. Then, by Tonelli's theorem.

$$< Tf, g > = \int_0^1 \Big(\int_0^1 k(x, y) f(y) dy \Big) \overline{g}(x) dx =$$

$$= \int_0^1 \overline{\Big(\int_0^1 \overline{k}(x, y) g(x) dx \Big)} f(y) dy = < f, T^*g > .$$

So,

$$(T^*g)(x) = \int_0^1 \overline{k}(y,x)g(y)dy.$$

(ii) Let $X = l^2$ and R be the right-shift $R(x_n) = (0, x_1, x_2, ...)$. Then R^* is the left-shift $L(x_n) = (x_2, x_3, ...)$.
(iii) Let $X = L_2(\mathbb{R})$ and $h : \mathbb{R} \to \mathbb{C}$ be a bounded measurable function. Define $M_h(x) = h(x)f(x)$ (multiplication operator). $M_h \in \mathcal{B}(X)$ and $(M_h)^* = M_{\overline{h}}$.

2.4 Self-adjoint operators

$T \in \mathcal{B}(X) = \mathcal{B}(X, X)$ is called a self-adjoint operator if $T^* = T$.

The above integral operator is self-adjoint if $k(x,y) = \overline{k}(y,x)$ for any $x, y \in]0,1[$.

The statement below is about norm of an operator in a Hilbert space.

Lemma 2.7. *Let X be a Hilbert space.*
(i) Let $T \in \mathcal{B}(X)$. Then

$$\|T\| = \sup\{| < Tx, y > | : \|x\| = \|y\| = 1\}.$$

(ii) Let $T \in \mathcal{B}(X)$ be self-adjoint. Then

$$\|T\| = \sup_{\|x\|=1} < Tx, x > .$$

Proof. We start with (i). By the definition of the operator norm,

$$\|T\| = \sup_{\|x\|=1} \|Tx\|.$$

We also know that

$$\|z\| = \sup_{\|y\|=1} | < z, y > |.$$

Then

$$\sup_{\|y\|=1} | < Tx, y > | = \|Tx\|.$$

The result follows.

(ii): Now, we let $K = \sup_{\|x\|=1} | < Tx, x > | \leq \|T\|$. Given $\varepsilon > 0$, one can find a pair x and y so that $\|y\| = \|x\| = 1$ and $\|T\| - \varepsilon < | < Tx, y > |$. Replacing y with $e^{i\theta} y$ does not change $\|y\|$ and $| < Tx, y > |$ but one can find θ so that $| < Tx, y > | = \text{Re} < Tx, y >$. Then

$$4(\|T\| - \varepsilon) \leq 4\text{Re} < Tx, y > = < T(x+y), (x+y) > - < T(x-y), (x-y) > \leq$$

$$\leq K(\|x+y\| + \|x-y\|) \leq 4K.$$

The required result follows. □

Proposition 2.8. *Let X be a Hilbert space and let $T \in \mathcal{B}(X)$. Then*
(i) $\|T^*T\| = \|T\|^2$.
(ii) If T is self-adjoint, $\|T^2\| = \|T\|^2$.

Proof. (i): We apply Lemma 2.7, noting that T^*T is self-adjoint as well,

$$\|T^*T\| = \sup_{\|x\|=1} < | < T^*Tx, x > | = \sup_{\|x\|=1} \|Tx\|^2 = \|T\|^2.$$

(ii) is obvious. □

2.5 Further elementary properties of self-adjoint operators

Let X, Y be Hilbert spaces, $S, T \in \mathcal{B}(X, Y)$.
(i) $(\lambda S + \mu T)^* = \overline{\lambda} S^* + \overline{\mu} T^*$.
(ii) $I_X^* = I_X$.
(iii) Assume $T \in \mathcal{B}(X)$. Then T is invertible if and only if T^* is invertible and in that case $(T^*)^{-1} = (T^{-1})^*$.

(i) and (ii) are entirely elementary. With regards to (iii), assume that T is invertible so that there exists $S = T^{-1} \in \mathcal{B}(X)$ such that $ST = TS = I_X$. Then $T^*S^* = S^*T^* = I_X^* = I_X$. This says that T^* has inverse, namely $S^* = (T^{-1})^*$.

The following results are similar to corresponding ones for dual operators, but easier to work with.

Proposition 2.9. *Let X, Y be Hilbert spaces and $T \in \mathcal{B}(X, Y)$. Then*
(i) $\text{Ker}\, T = (\text{Im}\, T^*)^\perp$*;*
(ii) $\text{Ker}\, T^* = (\text{Im}\, T)^\perp$*;*
(iii) $(\text{Ker}\, T^*)^\perp = \overline{\text{Im}\, T}$.

Proof. (i),(ii): the same as in finite-dimensional case. For (iii), use Corollary 1.6. □

Theorem 2.10. *Let X be a Hilbert space, $X = Y \oplus Z$, where Y and Z are both closed subspace of X. Let P be the operator $P(y+z) = y$. Then the following are equivalent:*
(i) $Z = Y^{\perp}$;
(ii) $P^* = P$;
(iii) $\|P\| \leq 1$. Moreover, for such a projector, $\|P\| = 1$ or $P = 0$.

Proof. Equivalence of (i) and (ii) as in finite dimensional case, so equivalence with (iii) is the interesting fact here, see Problem sheets Qn. 11. □

2.6 Unitary Operators

An operator between Hilbert spaces is unitary if it isometric and surjective.

Proposition 2.11. *Let $T, U : X \to Y$ be bounded linear operators between Hilbert spaces.*
(i) The following equivalent:
  (a) T is isometric;
  (b) $< Tx, Ty > = < x, y >$ for all $x, y \in X$;
  (c) $T^*T = I_X$.
(ii) The following is equivalent:
  (a) U is unitary;
  (b) $U^*U = I_X$ and $UU^* = I_Y$;
  (c) U and U^* are both isometric.

Proof. Exercise. □

Here are some example of unitary operators:
(i) The left shift is unitary on $l^2(\mathbb{Z})$.
(ii) A multiplication operator M_h is unitary on L_2 if and only if $|h| = 1$ a.e.
(iii) For measurable, non-negative g, the map $f \mapsto g^{\frac{1}{2}} f$ is isometric from $L_2(\mathbb{R}, gdt)$ to $L_2(\mathbb{R})$. It is unitary if and only if $g > 0$ a.e.

3 The Baire Category Theorem

3.1 Nowhere dense sets

Assume that M is a topological space, let us recall general topological notions:
(i) $G \subseteq M$ is dense if $M = \overline{G}$;
(ii) The interior of a subset S of M is

$$S^\circ = \cup(U \,\text{open} : U \subseteq S).$$

Recall that $\overline{M \setminus S} = M \setminus S^\circ$ and that, as a consequence, $M \setminus S$ is dense if and only if $S^\circ = \emptyset$.

Definition 3.1. *A subset E of M is nowhere dense if $(\overline{E})^\circ = \emptyset$.*

Lemma 3.2. *The following are equivalent:*
(i) E is nowhere dense:
(ii) $M \setminus \overline{E}$ is dense;
(iii) $M \setminus E$ contains a dense open set.

Proof. Assume E is nowhere dense but $M \setminus \overline{E}$ is not dense. Then there exists a closed set F such that $M \setminus \overline{E} \subseteq F \subset M$. So, $\overline{E} \supseteq M \setminus F$ and thus we get a contradiction. Implication $(i) \Rightarrow (ii)$ is proven and so is $(i) \Rightarrow (iii)$.

Now, assume that (ii) is true but there exists an non-empty open set O such that $(\overline{E})^\circ \supset O$. Then $M \setminus (\overline{E})^\circ \subseteq M \setminus O = F \subset M$. Since F is close, a contradiction follows and $(ii) \Rightarrow (i)$.

Suppose that (iii) is true, i.e., there exists an open set $O \subseteq M \setminus E$ and $\overline{O} = M$. Then, since $M \setminus O$ is closed and contains E, it contains $\overline{E}$ as well. So, O is a subset of $M \setminus \overline{E}$ and thus $M \setminus \overline{E}$ is dense. $(iii) \Rightarrow (ii)$ is proven. □

As an example of a nowhere dense set, consider the Cantor set in $[0, 1]$. From the construction, it follows that the Cantor set is closed but does not contain open subsets. Since the Cantor set has Lebesgue's measure zero, one may think that it is a necessary condition for a set to be nowhere dense. But it is not true. Moreover, using a similar construction, one can find a closed subset of $[0, 1]$ that contains no open subsets and has measure equal to $1 - \varepsilon$ for any given $0 < \varepsilon < 1$.

It is worthy to notice that although $\mathbb{Q} \subset \mathbb{R}$ has zero measure but it is not a nowhere dense set, since $\mathbb{Q}$ is dense in $\mathbb{R}$.

3.2 The Baire Category Theorem

In what follows we need Cantor's Nested Set Theorem.

Theorem 3.3. *Let M be a complete metric space and let $F_1 \supseteq F_2 \supseteq \dots$ be decreasing sequence of non-empty closed sets with* $\operatorname{diam} F_n \to 0$*. Then $\bigcap_n F_n$ is a singleton $\{x\}$.*

Proof. Pick $x_n \in F_n$. It is a Cauchy sequence, since, for $n > m$, $d(x_n, x_m) \le \operatorname{diam} F_m$. Let $x = \lim x_n$. Then $x \in \bigcap_n F_n$ since all F_n are closed. Diameter condition implies the intersection is a singleton. □

Definition 3.4. *A countable union of nowhere sets is said to be first category or meager.*

Theorem 3.5. *Let M be a complete metric space. The following are true and equivalent.*
(A) A countable intersection of dense open subsets of M is dense.
(B) The compliment of a countable union of nowhere sets in M is dense.
(C) A countable union of nowhere dense sets in M has empty interior.

Proof. We starts with the proof of equivalence.

Assume that (A) is true. Now, let E_k, $k = 1, 2, \dots$, be nowhere dense sets. We know that $M \setminus \overline{E}_k =: O_k$ is dense, see Lemma 3.2. So, sets O_k open and $\overline{O}_k = M$. By (A),

$$M = \overline{\bigcap_k O_k} = \overline{\bigcap_k (M \setminus \overline{E}_k)} = \overline{M \setminus \bigcup_k \overline{E}_k} \subseteq \overline{M \setminus \bigcup_k E_k} \subset M.$$

So, (A)⇒(B).

Now, assume that (B) is true. Consider open sets O_k, $k = 1, 2, \dots$, that are dense in M. Consider also sets $E_k = M \setminus O_k$. By Lemma 3.2, they are nowhere dense. Then by (B)

$$M = \overline{M \setminus \bigcup_k E_k} = \overline{\bigcap_k (M \setminus E_k)} = \overline{\bigcap_k O_k}$$

So, (B)⇒ (A). Now, take sets E_k that are nowhere dense. Suppose that there exists a non-empty open set $O \subseteq \bigcup_k E_k$. Then, since $M \setminus O$ is closed and by (B), $M \setminus O = \overline{M \setminus O} \supseteq \overline{M \setminus \bigcup_k E_k} = M$. This cannot be true. So, (B)⇒(C).

Now, assume that (C) is true. Take sets E_k that are nowhere dense. Assume that (B) is not true, i.e., there exists a proper closed subset F of M such that $M \setminus \bigcup E_k \subseteq F$. Therefore, $M \setminus F = O \subseteq \bigcup_k E_k$. Since O is open, we get a contradiction. So, (C)⇒(B) and the equivalence of (A), (B), (C) is proven.

Now, let us prove that (A) is true. Pick open sets O_k, $k = 1, 2, ...$, that are dense in M. We need to show that $\overline{\bigcap_k O_k} = M$. To this end, let us take an arbitrary open set O. It is clear that it is enough to prove

$$\bigcap_k O_k \cap O \neq \emptyset. \tag{3.1}$$

We are going to argue by induction. Since O_k is dense, $O_k \cap O' \neq \emptyset$ for each k and for any open set O'. Indeed, one may assume that $\overline{B}(x_1, r_1) \subset O \cap O_1$ for some $x_1 \in X$ and $0 < r_1 < 1$. Since $B(x_1, r_1) \cap O_2 \neq \emptyset$, we can find a point $x_2 \in X$ and $0 < r_2 < \min\{r_1, 1/2\}$ such that

$$\overline{B}(x_2, r_2) \subset B(x_1, r_1) \cap O_2 \subset O \cap O_1 \cap O_2.$$

And so on. As a result, we have a sequence

$$\overline{B}(x_n, r_n) \subset B(x_{n-1}, r_{n-1}) \cap O_n \subset O \cap O_1 \cap O_2 \cap ... \cap O_n$$

with $0 < r_n < \min\{r_{n-1}, 1/n\}$. By Cantor's Nested Set Theorem, applied to $F_n = \overline{B}(x_n, r_n)$, there exists a point $x \in O \cap O_1 \cap O_2 \cap ... \cap O_n$ for all n. Then (3.1) follows. □

It is worthy to mention the following famous example: the subset of $C([0, 1])$ of functions which are differentiable at some point is nowhere dense in $C([0, 1])$, see web sources for proof. This says that in a strong and precise sense 'most' real-valued continuous functions on $[0, 1]$ are nowhere differentiable.

4 Uniform Boundedness Theorem

4.1 Uniform Boundedness

Consider normed spaces X and Y and the space $\mathcal{B}(X, Y)$ of bounded (=continuous) linear maps from X to Y.

Let $\mathcal{E} \subseteq \mathcal{B}(X, Y)$. We say that $\mathcal{E}$ is point-wise bounded if for each $x \in X$ there exists a constant M_x such that $\|Tx\|_Y \le M_x$ for all $T \in \mathcal{E}$. We also say $\mathcal{E}$ is uniformly bounded if there exists a constant M such that $\|T\| \le M < \infty$ for all $T \in \mathcal{E}$. Obviously, uniform boundedness implies point-wise boundedness. The converse is the famous Banach-Steinhaus Theorem on Uniform Boundedness.

Theorem 4.1. *Let X be a Banach space, Y be a normed space, $\mathcal{E} \subseteq \mathcal{B}(X, Y)$ be such that, for each $x \in X$, $\sup_{T \in \mathcal{E}} \|Tx\|_Y < \infty$. Then $\sup_{T \in \mathcal{E}} \|T\| < \infty$.*

Proof. Consider sets $F_n = \{x \in X : \|Tx\|_Y \le n, \forall T \in \mathcal{E}\}$. Since T is continuous, F_n is closed for each n. Obviously, $X = \bigcup_n F_n$, at least some F_n is not nowhere dense, Theorem 3.5. Since F_n is closed, $F_n^\circ \neq \emptyset$ and thus there exists a non-empty ball $B(x_0, \delta) \subset F_n$. The latter means

$$\sup_{B(x_0,\delta)} \|Tx\|_Y \le n \quad \forall T \in \mathcal{E}.$$

Therefore, letting $z = x - x_0$, we find

$$\sup_{B(0,\delta)} \|Tz\|_Y - \|Tx_0\|_Y \le n$$

for the same T and

$$\sup_{B(0,\delta)} \|Tz\|_Y \le 2n$$

for the same T. One easily to conclude that $\|T\| \le 2n/\delta$ for all $T \in \mathcal{E}$. $\square$

The given proof of UBT is tailored to this particular result. The following Lemma synthesises the key ingredients, for potential use elsewhere.

Lemma 4.2. *Let X be a Banach space. Let C be a non-empty closed, convex subset of X such that $x \in C$ implies $-x \in C$ and $X = \bigcup_{n \ge 1} nC$. Then C contains a ball $B(0, \varepsilon)$ for some $\varepsilon > 0$.*

A proof of the lemma, and a derivation of UBT from it, is in fact Qn.15 on Problem Sheets.

4.2 Corollaries of UBT

Corollary 4.3. *Let X be a Hilbert space and $x^{(n)} \rightharpoonup x$. Then $\sup_{n\geq 1} \|x^{(n)}\| < \infty$.*

Proof. Define $T_n : X \to \mathbb{C}$ (or $\mathbb{R}$) by $T_n(y) =< y, x^{(n)} >$. It is easy to see that $\|T_n\| = \|x^{(n)}\|$. By assumptions, $sup_{n\geq 1}| < x^{(n)}, y > | < \infty$ for each fixed $y \in X$. The result follows from UBT. □

The following corollary of UBT is a good working example.

Corollary 4.4. *Let X be a real or complex Hilbert space and $\mathcal{E}$ be a subset of $\mathcal{B}(X)$ such that $\sup_{T\in\mathcal{E}} | < Tx, y > | < \infty$ for each $x, y \in X$. Then there is a constant C such that $\|T\| \leq C < \infty$ for all $T \in \mathcal{E}$.*

Proof. Let us fix $x \in X$. Define $K_{T,x} : X \to \mathbb{C}$ by $K_{T,x}(y) =< y, Tx >$. We know that $\|K_{T,x}\| = \|Tx\|$ and

$$\sup_{T\in\mathcal{E}} |K_{T,x}(y)| \leq \sup_{T\in\mathcal{E}} | < Tx, y > | < \infty$$

for each $x, y \in X$. By UBT, there exists a constant M_x such that

$$\sup_{T\in\mathcal{E}} \|Tx\| \leq M_x < \infty$$

for each fixed $x \in$. The result follows from UBT. □

4.3 Strong convergence of operators

Theorem 4.5. *Let X and Y be Banach spaces and a sequence $T_n \in \mathcal{B}(X, Y)$. The following are equivalent:*
(i) There exists $T \in \mathcal{B}(X, Y)$ such that $\lim_{n\to\infty} T_n x = Tx$ for all $x \in X$ (Strong convergence of a sequence T_n of operators to an operator T);
(ii) For each $x \in X$, $\lim_{n\to\infty} T_n x$ exists in Y;
(iii) There is a constant M and a dense subset Z of X such that
(a) $\|T_n\| \leq M$ for all n,
(b) $\lim_{n\to\infty} T_n x$ exists for each $x \in Z$.

Proof. $(i) \Rightarrow (ii)$ is obvious. To show $(ii) \Rightarrow (i)$, we observe that by the property of the limit, $T : X \to Y$ defined by $T(x) = \lim_{n\to\infty} T_n x$, is a linear map. Since $\|T_n x\|_Y \to \|Tx\|_Y$ and $\|Tx\|_Y \le \sup_{n\ge1} \|T_n x\|_Y$, we have $\|Tx\|_Y \le \sup_{n\ge1} \|T_n\|\|x\|_X$ for all $x \in X$. By UBT, $\sup_{n\ge1} \|T_n\| < \infty$ and $\|T\| \le \sup_{n\ge1} \|T_n\|$. So, $T \in \mathcal{B}(X, Y)$.

$(ii) \Rightarrow (iii)$ and $(i) \Rightarrow (iii)$ follow from UBT, see the above arguments.

Now, assume (iii) is true. We are going to use the density arguments, similar to those in the proof of Theorem 2.6. Indeed, take any $x \in X$ and $z \in Z$. Then we have

$$\|T_n x - T_m x\|_Y \le \|(T_n - T_m)z\|_Y + \|(T_n - T_m)(x - z)\|_Y \le$$

$$\le \|(T_n - T_m)z\|_Y + 2M\|x - z\|_Y$$

Given $x \in X$ and $\varepsilon > 0$, we find $z \in Z$ such that $2M\|x - z\|_Y < \varepsilon/2$ and then a number N such that $\|(T_n - T_m)z\|_Y < \varepsilon/2$ for $m, n > N$. So, $(iii) \Rightarrow (ii)$. Moreover, in the same way as in the proof of $(ii) \Rightarrow (i)$, we show that $\|T\| \le M$. Uniqueness follows from the identity $T_1 x - T_2 x = 0$ for any $x \in X$. □

4.4 Examples of strong convergence

1. Let L be the left-shift operator on l^1. Then $\lim_{n\to\infty} \|L^n x\|_{l^1} = 0$ for each $x \in l^1$, but $\|L^n\| = 1$ for any n and thus L^n does not converges to zero in $\mathcal{B}(X)$. This shows that the strong convergence of operators in general does not imply the uniform convergence of them.

2. Consider the shift operators T_h $(h \in \mathbb{R}^n)$ on $X = L_p(\mathbb{R}^n)$, where $1 \le p < \infty$:

$$(T_h)f(t) := f(t + h) \qquad (f \in L_p(\mathbb{R}^n),\ t \in \mathbb{R}^n).$$

Then $T_h \to I$ strongly as $h \to 0$, i.e.,

$$\|f(\cdot + h) - f(\cdot)\|_X = \Big(\int_{\mathbb{R}^n} |f(t + h) - f(t)|^p dt\Big)^{\frac{1}{p}} \to 0 \tag{4.1}$$

as $h \to 0$ for any $f \in X$. This property is known as integral continuity of functions from $L_p(\mathbb{R}^n)$. To see that it is true, consider a continuous function

g that is equal to zero outside a bounded set B_g of $\mathbb{R}^n$ (it is said that g is compactly supported in $\mathbb{R}^n$). By Lebesgue's theorem about dominated convergence

$$\int\limits_{\mathbb{R}} |g(t+h) - g(t)|^p dt \to 0$$

as $h \to 0$. Indeed, $t \mapsto g(t)$ and $t \mapsto g(t+h)$ are equal to zero in $\mathbb{R}^n \setminus B$ for some ball B and for any $|h| \leq 1$. So, $|g(t+h) - g(t)|^p \leq (2N)^p \chi_B(t)$ for all $t \in \mathbb{R}^n$ and $|h| \leq 1$, where $N = \sup_{\mathbb{R}^n} |g(t)|$. Let us denoted by Z the set of all continuous functions g with a compact support in $\mathbb{R}^N$ (for each $g \in \mathbb{Z}$, there exists a ball B_g such that $g = 0$ in $\mathbb{R}^n \setminus B_g$). In fact, we have

Theorem 4.6. *The set Z is dense in $L_p(\mathbb{R}^n)$ provided $1 \leq p < \infty$.*

This will follow from the following statements:

Lemma 4.7. *Let $A \subset \mathbb{R}^n$ and $\varrho(x, A) = \inf\limits_{z \in A} |x - z|$. Then $|\varrho(x, A) - \varrho(y, A)| \leq |x - y|$.*

Proof. We have for $z \in A$

$$\varrho(x, A) \leq |x - z| \leq |x - y| + |y - z|,$$

which implies $\varrho(x, A) \leq |x - y| + \varrho(y, A)$. Replacing x with y and y with x, we complete the proof. □

Lemma 4.8. *Let $1 \leq p < \infty$ and E be a bounded measurable set in $\mathbb{R}^n$. Given $\varepsilon > 0$, there exists a function $g \in Z$ such that $\|\chi_E - g\|_{L_p(E)} < \varepsilon$.*

Proof. Since E is bounded and measurable, there exist a closed set $F \subseteq E$ and a bounded open set $\mathcal{O} \supseteq E$ such that $|\mathcal{O} \setminus F| < (\varepsilon/2)^p$. We let

$$0 \leq g(x) := \frac{\varrho(x, \mathbb{R}^n \setminus \mathcal{O})}{\varrho(x, F) + \varrho(x, \mathbb{R}^n \setminus \mathcal{O})} \leq 1$$

for $x \in \mathbb{R}^n$. Obviously, g is continuous function in $\mathbb{R}^n$. And

$$\|\chi_E - g\|_{p,E} \leq 2|\mathcal{O} \setminus F|^{\frac{1}{p}} \leq \varepsilon.$$

□

Now, the statement of Theorem 4.6 follows from the fact for any $f \in L_p(\mathbb{R}^n)$ and any $\varepsilon > 0$, one can find finite number of disjoint bounded measurable sets E_j and numbers c_j such that $\|f - \sum_j c_j \chi_{E_j}\|_{L_p(\mathbb{R}^n)} < \varepsilon$, see Integration course.

So, now (4.1) follows from Theorem 4.5, taking $M = 1$ (explain why).

3. Here, we are going to discuss the approximation of unity in $L_p(\mathbb{R}^n)$ with $1 \le p < \infty$ by convolution. It is also called a mollification. We will see why later.

Let $f \in L_p(\mathbb{R}^n)$ and $g \in L_1(\mathbb{R}^n)$. Then the integral

$$(f * g)(s) := \int\limits_{\mathbb{R}^n} g(s-t) f(t) dt = \int\limits_{\mathbb{R}^n} f(s-t) g(t) dt$$

exists for a.a. $s \in \mathbb{R}^n$ and thus defines a function which is itself in $L_p(\mathbb{R}^n)$ and satisfies $\|g * f\|_p \le \|f\|_p \|g\|_1$. Indeed, by Hölder inequality, for $1 < p < \infty$ with $p' = p/(p-1)$ (p=1 is easy),

$$|(g * f)(s)| \le \Big(\int\limits_{\mathbb{R}^n} |g(s-t)| dt \Big)^{\frac{1}{p'}} \Big(\int\limits_{\mathbb{R}^n} |g(s-t)||f(t)|^p dt \Big)^{\frac{1}{p}} \le$$

$$\le \|g\|_1^{\frac{1}{p'}} \Big(\int\limits_{\mathbb{R}^n} |g(s-t)||f(t)|^p dt \Big)^{\frac{1}{p}}.$$

Hence, by Tonelli theorem,

$$\|g * f\|_p^p \le \|g\|_1^{\frac{p}{p'}} \int\limits_{\mathbb{R}^n} \Big(\int\limits_{\mathbb{R}^n} |g(s-t)||f(t)|^p dt \Big) ds =$$

$$= \|g\|_1^{\frac{p}{p'}} \int\limits_{\mathbb{R}^n} \Big(\int\limits_{\mathbb{R}^n} |g(s-t)||f(t)|^p ds \Big) dt = \|g\|_1^p \|f\|_p^p.$$

Let us fix non-negative $g \in L_1(\mathbb{R}^n)$, with $\int\limits_{\mathbb{R}^n} g dt = 1$. In addition, we assume that $g = 0$ outside the unit ball B of $\mathbb{R}^n$, centred at zero. Now, define $g_j(t) = jg(jt)$. Then $\int\limits_{\mathbb{R}^n} g_j dt = 1$ and $\|g_n\|_1 = \|g\|_1 = 1$. Define $T_j : X = L_p(\mathbb{R}^n) \to X$ by $T_j(f) = g_j * f$. We wish to show that $T_j \to I$

strongly as $j \to \infty$. Repeating more or less the same arguments as above, we find

$$|(g_j * f)(s) - f(s)| = \Big| \int\limits_{\mathbb{R}^n} g_j(t)(f(t-s) - f(s))dt \Big|$$

$$\leq \Big(\int\limits_{\mathbb{R}^n} g_j(t)|f(t-s) - f(s)|^p dt \Big)^{\frac{1}{p}}$$

and thus after application of Tonelli's theorem

$$\|g_j * f - f\|_p^p \leq \Big(\int\limits_{\mathbb{R}^n} g_j(t) \Big(\int\limits_{\mathbb{R}^n} |f(s-t) - f(s)|^p ds \Big) dt.$$

Making the change of variable $t = \tau/j$, we get

$$\|g_j * f - f\|_p^p \leq \Big(\int\limits_{B} g(\tau) \Big(\int\limits_{\mathbb{R}^n} |f(s - \tau/j) - f(s)|^p ds \Big) dt.$$

By integral continuity, given $\varepsilon > 0$, we can find N such that, for any $|\tau| \leq 1$,

$$\|f(\cdot - \tau/j) - f(\cdot)\|_p \leq \varepsilon$$

as $j > N$. Hence, $\|g_j * f - f\|_p \leq \varepsilon$ for $j > N$.

It is worthy to know that if g is a smooth function, for each n the convolution $g_j * f$ has the same smoothness as function g for any $f \in L_p(\mathbb{R}^n)$.

5 Open Mapping Theorem and its applications

Let $f : X \to Y$, with X and Y any topological spaces. A map f is open if f maps open sets onto open sets. Note that, for a bijective map $f : X \to Y$, the well-defined map $f^{-1} : Y \to X$ is continuous if and only if f is open.

Lemma 5.1. *Let X and Y be normed spaces and $T : X \to Y$ be linear. The following are equivalent:*
(i) T open;
(ii) $T(B_X)$ open ($B_X = B_X(0, 1)$);
(iii) $T(B_X)$ contains some non-empty ball $B_Y(0, \varepsilon)$

Proof. Exercise. □

Theorem 5.2. *Let X and Y be Banach spaces and T be a bounded linear mapping of X onto Y. Then T is open.*

Proof. Let B_X and B_Y be the unit balls in X and Y.
Step 1 Here, we wish to show that if $T(X) = Y$ and X is normed, Y is Banach, then $\overline{T(B_X)} \supset B_Y(0, \varepsilon)$ for some positive ε. Indeed, it is an exercise to show that $\overline{T(B_X)}$ is a non-empty closed convex set having the property: $y \in \overline{T(B_X)}$ implies $-y \in \overline{T(B_X)}$. Since

$$T(X) = Y = \bigcup_n nT(B_X) \subseteq \bigcup_n n\overline{T(B_X)} \subseteq Y,$$

the statement follows from Lemma 4.2.
Step 2 Now, let us assume that $T : X \to Y$ is linear and continuous, X is Banach and Y is normed, and $\overline{T(B_X)} \supset B_Y(0, \varepsilon)$ for some positive ε. Our aim is to show that there exists $\delta > 0$ such that $T(B_X) \supset B_Y(0, \delta)$.

Let $y \in \overline{T(B_X)}$, then by assumption we can find $y_1 \in T(B_X)$ such that $\|y - y_1\|_Y < \varepsilon/2$. Since $y - y_1 \in B_Y(0, \varepsilon/2)$, then $y - y_1 \in \overline{T(\frac{1}{2}B_X)}$ and thus we can find $y_2 \in T(\frac{1}{2}B_X)$ such that $\|y - y_1 - y_2\|_Y < \varepsilon/4$. Proceeding in this way, we find sequences y_n and x_n with the following properties:

$$\|y - y_1 - y_2 - ... - y_i\|_Y < \varepsilon/2^i, \quad y_i = Tx_i, \quad x_i \in \frac{1}{2^{i-1}}B_X, \quad , i = 1, 2, ...$$

Since $\|x_i\|_X \leq \frac{1}{2^{i-1}}$ and X is Banach, the sequence $z_n = \sum\limits_{i=1}^{n} x_i$ converges to z in X. Moreover, $\|z_n\|_X \leq \sum_{i=1}^n 1/2^{i-1} \leq 2$. Hence, we have $\|y - Tz_n\|_Y < \varepsilon/2^n$ and passing to the limit we show $y = Tz$ with $z \in 3B_X$. This means that $T(3B_X) \supset B_Y(0, \varepsilon)$ So, we can take $\delta = \varepsilon/3$.

Now, the result follows from the statement of Step 2 and Lemma 5.1, see (iii). □

A direct corollary of OMT is the following theorem called Inverse Mapping Theorem or Banach's Isomorphism Theorem.

Theorem 5.3. *Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear bijection. Then T^{-1} is continuous.*

Now, our aim is to discuss applications of IMT for adjoint operators and closed images. Recall that for $T \in \mathcal{B}(X)$ (X is Hilbert space), the image TX will not in general be closed.

Theorem 5.4. *Let X, Y be Hilbert spaces and $T \in \mathcal{B}(X,Y)$. Then TX is closed if and only if T^*Y is closed.*

Proof. Suppose that T^*Y is closed in X. We let $N := \overline{TX} \subseteq Y$. Define, further, $S : X \to N$ by $Sx = Tx$ for $x \in X$. We know that there exists the adjoint operator $S^* : N \to X$. By Proposition 2.9, since $\overline{\operatorname{Im} S} = N = (\operatorname{Ker} S^*)^\perp$, S^* is injective. Let $P : Y \to N$ be a projection.

Let us prove that $S^*(Py) = T^*(y)$ for $y \in Y$. Indeed,

$$< Tx, y >_Y = < Sx, y >_Y = < Sx, Py >_Y = < x, S^*Py >_X = < x, T^*y >_X .$$

It follows from the above that: $S^*P = T^*$, $S^*N = T^*Y =: M$, and $S^*n = T^*n$ for all $n \in N$. So, we can introduce $V : N \to M$ by $Vn = S^*n$ for all $n \in N$. So, now V is a bijection, M, N are Banach spaces. By IMT, there exists continuous $V^{-1} : M \to N$. Just by simple properties of adjoint operators, we get $(V^*)^{-1} = (V^{-1})^* \in \mathcal{B}(N, M)$.

Now, take any $y \in Y$ and any $n \in N$, we know that there exists a unique $m \in M$ such that $n = V^*m$. So, we have

$$< n, y >_Y = < V^*m, y >_Y = < V^*m, Py >_Y = < m, V^{**}Py >_X =$$

$$= < m, VPy >_X = < m, S^*Py >_X = < Sm, Py >_Y =$$

$$= < Sm, y >_Y = < Tm, y >_Y$$

for $y \in Y$. Hence, $n = Tm$ and $N \subseteq TX$. So, TX is closed in Y.

Converse can be proved by observation that $T^{**} = T$ □

Now, let X and Y be Banach spaces, $T : X \to Y$ be a mapping. The set $G(T) := \{(x, y) : \ x \in X, \ y = T(x)\}$ is called the graph of T.

Define the norm $\|(x, y)\|_{X \times Y}$ on $X \times Y$ in one of the following ways: either $\|x\|_X + \|y\|_Y$, or $\sqrt{\|x\|_X^2 + \|y\|_Y^2}$, or $\max\{\|x\|_X, \|y\|_Y\}$. It is easy to see that if T is continuous then $G(T)$ is closed in $X \times Y$. Inverse in general is not true. However, we have (Closed Graph Theorem)

Theorem 5.5. *Let $T : X \to Y$ is a linear mapping with closed graph. Then T is continuous (bounded).*

Proof. By assumptions, $G(T)$ itself is a Banach space as a closed subset of a Banach space $X \times Y$. Let us introduce two operators $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. Obviously, $\pi_1 \in \mathcal{B}(X \times Y, X)$ and $\pi_2 \in \mathcal{B}(X \times Y, Y)$. Introduce $p = \pi_1|_{G(T)}$ $(p(x, Tx) = x)$. It is easy to see that operator p is one-to-one and and onto. Moreover, it is a continuous operator. By IMT, there exists p^{-1} and it is a continuous operator. On the other hand, $T(x) = \pi_2(p^{-1}x)$. The result follows. □

Let us discuss further applications of our main theorems.

1.

Proposition 5.6. *Let X be a Hilbert space and let $T : X \to X$ be a linear mapping. If $< Tx, y >=< x, Ty >$ for all x and y then T is bounded (and thus self-afoint).*

Proof. Suppose that $\|x - x_n\| \to 0$ and $\|Tx_n - z\| \to 0$. then for any $y \in X$ we have

$$< z, y >= \lim_{n\to\infty} < Tx_n, y >= \lim_{n\to\infty} < x_n, Ty >=< x, Ty >=< Tx, y > .$$

So, $z = Tx$ and thus $G(T)$ is closed and, by CGT, T is bounded. □

2. Let us discuss continuity of projection on a Banach space. Let X be a Banach space, Y and Z be its closed subspaces such that $X = Y \oplus Z$. Define $P : X \to X$ by $P(y + z) = y$. Let us show that P is continuous. Let $\|x_n - x\| \to 0$ and $\|Px_n - y\| \to 0$. We let $y_n = Px_n \in Y$, and $z_n = x_n - y_n \in Z$. Obviously, $y \in Y$ and $z_n \to z = x - y \in Z$ in X. By uniqueness of decomposition, $y = Px$. The result follows from CGT.

3. IMT has a simple application to a proof of equivalence norms.

Proposition 5.7. *Let X be a Banach space with respect to two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ and suppose that there exists a constant c such that $\|x\|_1 \le c\|x\|_2$ for all $x \in X$. Then there exists a constant c' such that $\|x\|_2 \le c'\|x\|_1$.*

Proof. This is an immediate consequence of IMT, applied to the Identity map $id(X, \|\cdot\|_2) \to (X, \|\cdot\|_1)$. This map is continuous by hypothesis and a linear bijection. Hence, id^{-1} is continuous too. □

4. Now, consider a multiplicative operator on $L_1(\mathbb{R})$. Let h be a measurable function such that $fh \in L_1(\mathbb{R})$ for each $f \in L_1(\mathbb{R})$ and consider $M_h : f \to fh$.

Then M_h is well defined as a map from $L_1(\mathbb{R})$ to itself. We claim that M_h has the closed graph and so, by CGT, is continuous.

To prove this assume $f_n \to f$ and $f_n h \to g$ in $L_1(\mathbb{R})$. Then by passing to a suitable subsequence, we may assume that that convergence occurs pointwise a.e. But then $f_n h \to fh$ a.e., so $g = fh$ a.e. and thus $g = M_h f$.

Boundedness of M_h implies that there exists a constant K such that $\int |fh|dx \leq K \int |f|dx$ for all $f \in L_1(\mathbb{R})$. We claim that there exists a Lebesgue measure zero set N such that h is bounded on $\mathbb{R} \setminus N$. Suppose this fails. Then $E = \{x : \ |h(x)| > 2K\}$ is a non-zero measure set. This implies that there exists n such that $|E \cap [-n,n]| \neq 0$. Considering $f = \chi_{E\cap[-n,n]}$, we get a contradiction.

Now, let us discuss how IMT and CGT conclusions can fail if not all the conditions from them are met.

1. Consider the map $T : X \subset l_1 \to l_1$ defined by

$$T(x_j) = (jx_j)$$

with

$$X = \{x = (x_j) \in l_1 : \ \sum_{j=1}^{\infty} |jx_j| < \infty\}.$$

Obviously, X is a proper subset of l_1 since $(1/j^2) \in l_1 \setminus X$. X is dense (consider sequences with only finitely many non-zero coordinates) and thus X is not complete with respect to l_1-norm. Moreover, T is not bounded since $T(e^{(n)}) = n$. However, $G(T)$ is closed in $l_1 \times l_1$, which can be verified directly using coordinate convergence. Conclusion is that CGT does now works since X is not a Banach space with l_1-norm.

2. Define the operator $T : X \to C([0,1])$ by $Tf = f'$, where X is the subspace $C^1([0,1])$ of $C([0,1])$ consisting of those f for which $f' \in C([0,1])$. X is a proper dense subset of $C([0,1])$. The operator T is not bounded as there exists a sequence (f_n) in X such that $\|f_n\|_\infty = 1$ and $\|f'_n\|_\infty$ is unbounded. Nonetheless, $G(T)$ is closed in $C([0,1]) \times C([0,1])$. Indeed, let us $g_n \to g$ and $g'_n \to G$ in $C([0,1])$. It is easy to see that limit functions satisfy the following identity

$$g(t) - g(s) = \int_s^t G(\tau)d\tau$$

for all $s,t \in [0,1]$. This implies that $G = f'$.

6 Spectral Theory in Hilbert Space

In this section always assume that the scalar field is $\mathbb{C}$.

6.1 Main definitions

We start with some definition and results from B4.1 that we are going to use in this course.

Definition 6.1. *Let X be a complex Banach space and $T \in \mathcal{B}(X)$. Spectrum of T is the set*

$$\sigma(T) := \{\lambda \in \mathbb{C} : \ (\lambda I - T) \textit{ has no inverse in } \mathcal{B}(X)\}.$$

It is known that $\sigma(T)$ is non-empty, closed set of $\mathbb{C}$ and if $\lambda \in \sigma(T)$ then $|\lambda| \leq \|T\|$.

Denote by $\sigma_p(T)$ the set of $\lambda \in \mathbb{C}$ such that $\operatorname{Ker}(\lambda I - T)$ contains non-zero vectors called eigenvectors belonging to the eigenvalue λ.

It is said $\lambda \in \mathbb{C}$ is an approximate eigenvalue if there exists a sequence $x_n \in X$ such that $\|x_n\| = 1$ and $\|Tx_n - \lambda x_n\| \to 0$ as $n \to \infty$. The set of all approximate eigenvalues is denoted by $\sigma_{ap}(T)$. It is also known from B4.1 that

$$\sigma_p(T) \subseteq \sigma_{ap}(T) \subseteq \sigma(T). \tag{6.1}$$

Proposition 6.2. *Let X be a Banach space and $S \in \mathcal{B}(X)$. Assume that there exists a positive number δ such that $\|Sx\| \geq \delta\|x\|$ for all $x \in X$. Then S is injective and SX is closed. Moreover, if in addition, we assume that SX is dense, then S is surjective and has inverse $S^{-1} \in \mathcal{B}(X)$, with $\|S^{-1}\| \leq 1/\delta$.*

The next statement is easy consequence of Proposition 2.9.

Proposition 6.3. *Let X be a complex Hilbert space, $T \in \mathcal{B}(X)$ and $\lambda \in \mathbb{C}$. Then*
(i) $(\lambda I - T)^* = \overline{\lambda} I - T^*$;
(ii) $\lambda I - T$ *is invertible if and only if* $\overline{\lambda} I - T^*$ *is invertible;*
(iii) $\operatorname{Ker}(\lambda I - T) = \operatorname{Im}(\overline{\lambda} I - T^*)^{\perp}$;
(iv) $\operatorname{Ker}(\overline{\lambda} I - T^*)^{\perp} = \overline{\operatorname{Im}(\lambda I - T)}$.

Let us consider an example on the last few results.

Proposition 6.4. *Let X be a complex Hilbert space and let $T \in \mathcal{B}(X)$, T is self-adjoint. Then $iI + T$ has an inverse $(iI+T)^{-1} \in \mathcal{B}(X)$. Moreover, $(iI+T)^{-1}(iI-T)$ is unitary.*

Proof. Let $S = iI + T$ and $R = iI - T$. We have

$$\|(iI \pm T)x\|^2 = \pm i < Tx, x > + < Tx, Tx > \mp i < x, Tx > + < x, x >=$$

$$\|Tx\|^2 + \|x\|^2 \geq \|x\|^2.$$

Now, apply Proposition 6.2 with $iI \pm T$ in place of T (and $\delta = 1$) to get S and R both injective and SX and RX closed. Moreover, $S^* = -R$ and $R^* = -S$. Then, by Proposition 2.9,

$$(SX)^{\perp} = \operatorname{Ker} S^* = \operatorname{Ker} R = \{0\}, \quad (RX)^{\perp} = \operatorname{Ker} R^* = \operatorname{Ker} S = \{0\}.$$

So, both S and R are invertible.

Next we let $U = S^{-1}R$. U is invertible and $U^{-1} = R^{-1}S$. We then have

$$U^* = R^*(S^{-1})^* = R^*(S^*)^{-1} = SR^{-1}.$$

Since R and S and thus R^{-1} and S^{-1} commute, then, for example,

$$U^*U = (SR^{-1})(S^{-1}R) = I.$$

The result follows. □

6.2 Adjoints and spectra

Proposition 6.5. *Let X be a Hilbert space, $T \in \mathcal{B}(X)$, and $\lambda \in \mathbb{C}$. Then*
(i) $\lambda \in \sigma(T)$ if and only if $\overline{\lambda} \in \sigma(T^*)$;
(ii) If T is normal, i.e., $TT^* = T^*T$, then $\operatorname{Ker}(\lambda I - T) = \operatorname{Ker}(\overline{\lambda} I - T^*)$;
(iii) If T is self-adjoint then $\sigma_p(T) \in \mathbb{R}$. Eigenvectors belonging to different eigenvalues are orthogonal.

Proof. (i) come from Proposition 6.3. For (ii), we notice that $\lambda I - T$ is normal as well and that $\|Sx\| = \|S^*x\|$ for any $x \in X$ provided S is normal. (iii) is proved as in the finite dimensional case. □

The next theorem can be deduced from the course B4.1 but let us give an independent proof using the notion of weak convergence.

Theorem 6.6. *Let X be a Hilbert space and $T \in \mathcal{B}(X)$. Then*

$$\sigma(T) = \sigma_{ap} \cup \sigma_p'(T^*),$$

where $\lambda \in \sigma_p'(T^*)$ if and only if $\overline{\lambda} \in \sigma_p(T^*)$.

Proof. Assume that $\lambda \in \sigma(T)$. Since $\sigma_p(T) \subseteq \sigma_{ap}(T)$, we need to consider only the case when $\mathrm{Ker}\,(\lambda I - T) = \{0\}$, i.e., the operator $\lambda I - T$ is injective. Then, there are only two options either $\overline{\mathrm{Im}\,(\lambda I - T)}$ is proper closed subset of X or $\overline{\mathrm{Im}\,(\lambda I - T)} = X$ In the first case, by Proposition 6.3, see (iv), $\mathrm{Ker}\,(\overline{\lambda} I - T^*) \subset X$ and $\overline{\lambda}$ is an eigenvalue of T^*, i.e., $\overline{\lambda} \in \sigma_p(T^*)$ and thus $\lambda \in \sigma_p'(T^*)$. In the second case, one find $y \in X \setminus \mathrm{Im}\,(\lambda I - T)$ and a sequence $y_n \in \mathrm{Im}\,(\lambda I - T)$ such that $y_n \to y$ in X. Let $y_n = (\lambda I - T)x_n$. Let us show that the sequence x_n is unbounded. Indeed, if it is bounded, there exists a subsequence x_{n_k} weakly converging to x. Now, for any $z \in X$, we have

$$< y_{n_k}, z >=< (\lambda I - T)x_{n_k}, z >=< x_{n_k}, (\overline{\lambda} I - T^*)z > .$$

Passing to the limit as $k \to \infty$, we show

$$< y, z >=< x, (\overline{\lambda} I - T^*)z >=< (\lambda I - T)x, z > .$$

The latter implies $y \in \mathrm{Im}\,(\lambda I - T)$, which is wrong, and thus sequence x_n is unbounded. Hence, there exists a subsequence $\|x_{n_k}\| \to \infty$. Then, taking $z_{n_k} = x_{n_k}/\|x_{n_k}\|$, we have $\|(\lambda I - T)(z_{n_k})\| = \|y_{n_k}\|/\|x_{n_k}\| \to 0$ with $\|z_{n_k}\| = 1$, which means that $\lambda \in \sigma_{ap}(T)$.

So, we have proven $\sigma(T) \subseteq \sigma_{ap}(T) \cup \sigma_p'(T^*)$. The opposite inclusion is easy, see (6.1) and Proposition 6.3 (i). □

Now, we can prove the following.

Theorem 6.7. *Let X be a complex Hilbert space and $T \in \mathcal{B}(X)$ be self-adjoint. Then $\sigma(T) \in \mathbb{R}$.*

Proof. By the previous theorem and by Proposition 6.5, the spectrum of a self-adjoint operator T is reduced to $\sigma_{ap}(T)$. So, let λ is an approximate eigenvalue, i.e., there exists a sequence x_n such that $\|x_n\| = 1$ and $\|(\lambda I - T)x_n\| \to 0$. Then

$$\overline{\lambda} - \lambda = (\overline{\lambda} - \lambda) < x_n, x_n >=$$

$$=< Tx_n - \lambda x_n, x_n > - < Tx_n - \overline{\lambda} x_n, x_n >=$$

$$=< Tx_n - \lambda x_n, x_n > - < x_n, T^* x_n - \lambda x_n >=$$

$$=< Tx_n - \lambda x_n, x_n > - < x_n, Tx_n - \lambda x_n > \to 0.$$

□

We know that, for a self-adjoint operator, acting on a complex Hilbert space, $< Tx, x >$ is real. We also know that $|\lambda| \leq \|T\|$ for $x \in \sigma(T)$ and that $\|T\| = \sup\{| < Tx, x > | : \ \|x\| = 1\}$.

Lemma 6.8. *Let X be a complex Hilbert space and $T \in \mathcal{B}(T)$ be self-adjoint. Assume that a unit vector x_0 satisfies the condition $\|T\| = | < Tx_0, x_0 > |$. Then x_0 is an eigenvector of T belonging to a eigenvalue λ_0, i.e., $Tx_0 = \lambda_0 x_0$, such that $\|T\| = |\lambda_0|$.*

Proof. Assume that $< Tx_0, x_0 >$ is non-negative. Pick any $y \in Y$ such that $< y, x_0 > 0$ and consider a vector $x = (x_0 + \alpha y)/\sqrt{1 + |\alpha|^2 \|y\|^2}$ for any $\alpha \in \mathbb{C}$. Obviously, $\|x\| = 1$. Simple calculations show

$$< Tx, x >= \frac{1}{1 + |\alpha|^2 \|y\|^2} (< Tx_0, x_0 > + < Tx_0, \alpha y > + < T(\alpha y), x_0 > +$$

$$+|\alpha|^2 < Ty, y >) =< Tx_0, x_0 > + \overline{\alpha} < Tx_0, y > + \alpha \overline{< Tx_0, y >} +$$

$$+|\alpha|^2 < Ty, y > \leq < Tx_0, x_0 > .$$

The latter inequality implies $< Tx_0, y >= 0$ (explain why).

Now, let $Y = \text{span}\,\{x_0\}$ and then $X = Y \oplus Y^\perp$. We have proven $Tx_0 \in (Y^\perp)^\perp = Y$. Hence, $Tx_0 = \lambda_0 x_0$ and $< Tx_0, x_0 >= \lambda_0$. □

Proposition 6.9. *Let X be a Hilbert space and $U \in \mathcal{B}(X)$ be a unitary. If $\lambda \in \sigma(U)$, then $|\lambda| = 1$.*

Proof. We have proved that U is surjective isometry and U^{-1} exists and equal U^*. Indeed, $|\lambda| \leq \|U\| = 1$. Assume for contradiction that there exists λ with $|\lambda| < 1$ and $\lambda I - U$ is not invertible. Then $\overline{\lambda} I - U^*$ is not invertible. This implies that $(\overline{\lambda} I - U^*)U = \overline{\lambda} U - I$ is not invertible either. This a contradiction. □

6.3 Example: spectra

1.Consider

$$T(\alpha_j) = (\alpha_j/j)$$

on $l_2(\mathbb{N})$. Then $\sigma(T) = \{k^{-1} : \ 1, 2, ...\} \cup \{0\}$.
2. Consider a complex Hilbert $l_2(\mathbb{Z})$ with inner product

$$< (\alpha_j), (\beta_j) >= \sum_{j=-\infty}^{\infty} \alpha_j \overline{\beta}_j.$$

Let U be given by $U\alpha = \beta$ where $\beta_j = \alpha_{j-1}$. Then U is unitary and

$$\sigma_p(U) = \emptyset, \qquad \sigma(T) = \{\lambda : \ |\lambda| = 1\},$$

see Problem sheet, Qn 22.
3. Let T be the operator on the complex Hilbert space $L_2(0,1)$ given by

$$Tf(t) = tf(t).$$

Then T is self-adjoint and $\sigma(T) = [0,1]$ and $\sigma_p(T) = \emptyset$.
4.

Definition 6.10. *Let X be Y be two normed spaces. $K \in \mathcal{B}(X,Y)$ is a compact operator if, for any bounded set $B \in X$, $K(B)$ is a precompact set in Y. (Recall from Part A: a set $\mathcal{K}$ of a metric space $\mathcal{M}$ is called precompact set if any sequence of $\mathcal{K}$ contains a subsequence converging in $\mathcal{M}$).*

Simple examples:
(a) if $\dim Y < \infty$, then any $K \in \mathcal{B}(X,Y)$ is precompact.
(b) Let $X = Y = L_2(0,1)$ and

$$Kf(t) := \int_0^t f(s)ds$$

for $f \in X$. Now, let f_j is a bounded sequence of X. We can select a subsequence such that $f_{j_m} \rightharpoonup f$ in X. Denoting by χ_t the characteristic function of the interval $]0,t[$, we have (due to the weak convergence)

$$Kf_{j_m}(t) = \int_0^1 f_{j_m}(s)\chi_t(s)ds \to \int_0^1 f(s)\chi_t(s)ds = Kf(t)$$

for all $0 < t < 1$. On the other hand, the Cauchy-Schwarz inequality implies:

$$\sup_{0<t<1} |Kf_{j_m}(t)| \leq \sqrt{t}\|f_{j_m}\|_X \leq \sup_j \|f_j\|_X < \infty.$$

Then, the fact

$$\int_0^1 |Kf_{j_m}(t) - Kf(t)|^2 dt \to 0$$

as $m \to \infty$ follows from the Dominated Convergence Theorem.

Let $K \in \mathcal{B}(X)$ be a compact operator in infinite dimensional space X. Then $0 \in \sigma(K)$. Indeed, assume that there exists $K^{-1} \in \mathcal{B}(X)$. Let B be the unit ball of X. Since $K^{-1}(B)$ is bounded, the set $B = K(K^{-1}B)$ must be precompact which is not true.

There are several nice properties of compact self-adjoint operators.

Proposition 6.11. *Let K be a compact self-adjoint operator on a complex Hilbert space X. If $\lambda \in \sigma(T)$ and $\lambda \neq 0$, then $\lambda \in \sigma_p(K)$.*

Proof. Since K is a self-adjoint, λ is an approximate eigenvalue, i.e., there exists a sequence $x_n \in X$ such that

$$\|x_n\| = 1, \qquad \|Kx_n - \lambda x_n\| \to 0.$$

We know that there exists a subsequence $x_{n_j} \rightharpoonup x$. Since K is compact $Kx_{n_j} \to Kx$ in X (explain why) and thus (since $\lambda \neq 0$) $x_{n_j} \to x$. Obviously, $\|x\| = 1$, $Kx - \lambda x = 0$. □

Theorem 6.12. *Let K be a compact self-adjoint operator on a complex Hilbert space X. Then there exists at least one eigenvector of K. Moreover, this eigenvector belongs to an eigenvalue λ_0 of K satisfying $\|K\| = |\lambda_0|$.*

Proof. Without loss of generality, we may assume that $K \neq 0$. We know that for any self-adjoint operator K

$$\|K\| = \sup\{| < Kx, x > | : \ \|x\| = 1\},$$

see Lemma 2.7. Let a sequence x_n be such that $\|x_n\| = 1$ and $| < Kx_n, x_n > | \to \|K\|$. We also can find a subsequence $x_{n_j} \rightharpoonup x_0$ with

$$1 = \liminf_{j\to\infty} \|x_{n_j}\| \geq \|x_0\|,$$

see Proposition 2.4, and, since K is a compact operator, $Kx_{n_j} \to Kx_0$ and thus $< Kx_{n_j}, x_{n_j} > \to < Kx_0, x_0 > = \|K\|$ (explain why). Clearly, $x_0 \neq 0$ and in fact, $\|x_0\| = 1$. If not, let $x' = x_0/\|x_0\|$, then $\|x'\| = 1$ and

$$| < Kx', x' > | = \frac{1}{\|x_0\|^2}\|K\| \leq \|K\|,$$

which is wrong if $\|x_0\| < 1$. The result follows from Lemma 6.8. □

Theorem 6.13. *Let K be a compact on a complex Hilbert space X. Let $\delta > 0$ and introduce the set*

$$\Sigma = \text{span}\,\{x \in X : \ \|x\| = 1, \, Kx = \lambda x, \, |\lambda| \geq \delta\}.$$

Then $\dim \Sigma < \infty$.

Proof. Assume for contradiction that for any n there exist linearly independent vectors x_1, x_2,...,x_n such that $x_i \neq 0$, $Kx_i = \lambda_i x_i$, with $|\lambda_i| \geq \delta$, $i = 1, 2, ..., n$. We let $E_n = \text{span}\,\{x_1, x_2, ..., x_n\}$. By construction, E_{n-1} is a proper subspace E_n.

Now, let $y_1 = x_1/\|x_1\|$. Our aim is to show that there exists a sequence y_2, y_3,..., with the following properties: $y_n \in E_n$, $\|y_n\| = 1$, and $\text{dist}\,(y_n, E_{n-1}) > 1/2$, $n = 2, 3, ...$ Indeed, by assumptions $\text{dist}\,(x_n, E_{n-1}) = \alpha > 0$. Obviously, there exists $x_* \in E_{n-1}$ such that $\|x_n - x_*\| < 2\alpha$. Since $\alpha = \text{dist}\,(x_n - x_*, E_{n-1})$, we can let $y_n = (x_n - x_*)/\|x_n - x_*\|$. We then have $\|y_n\| = 1$, $y_n \in E_n$, and $\text{dist}\,(y_n, E_{n-1}) = \alpha/\|x_n - x_*\| > 1/2$. Notice that

$$\|y_n/\lambda_n\| \leq 1/\delta.$$

If we can show that sequence $K(y_n/\lambda_n)$ is not precompact, we get a contradiction.

So, we have

$$y_n = \sum_{j=1}^{n} \alpha_i x_i, \quad K(y_n/\lambda_n) = \sum_{j=1}^{n} \lambda_i \alpha_i x_i/\lambda_n = y_n + z_n,$$

where

$$z_n = \sum_{j=1}^{n-1} (\lambda_i/\lambda_n - 1)\alpha_i x_i \in E_{n-1}.$$

For any $n > m$, $z_n - y_m - z_m \in E_{n-1}$ and thus

$$\|K(y_n/\lambda_n) - K(y_m/\lambda_m)\| = \|y_n + z_n - y_n - z_m\| > 1/2.$$

So, there is no converging subsequence of $K(y_n/\lambda_n)$. □

7 Orthogonal sequences in Hilbert space

7.1 Orthonormal sequences

Let S be a non-empty set of vectors in an IPS X. Then
(i) S is orthogonal if $x, y \in S$ and $x \neq y$, then $< x, y >= 0$;
(ii) S is orthogonal if it orthogonal and in addition $\|x\| = 1$ for all $x \in S$.

An orthogonal set of non-zero vectors can be converted to orthonormal set be replacing each member x by $x/\|x\|$.

Proposition 7.1. *Let X be a Hilbert space, and let e_n, $n = 1, 2, ...$, be an orthonormal sequence in X. Then $\sum_{n=1}^{\infty} \alpha_n e_n$ converges in X if and only if $(\alpha_n) \in l^2$.*

Proof. Exerscise. □

Proposition 7.2. *Let X be a Hilbert space, and let $\{e_n\}$ be an orthonormal sequence X. Let $x \in X$ and $Y = \overline{\operatorname{span}\{e_n\}}$.*
(i) $\|x - \sum_{n=1}^{m} < x, e_n > e_n\|^2 = \|x\|^2 - \sum_{n=1}^{m} | < x, e_n > |^2$;
(ii) $\sum_{n=1}^{\infty} | < x, e_n > |^2 \leq \|x\|^2$ (Bessel's inequality);
(iii) $\sum_{n=1}^{\infty} < x, e_n > e_n = P_Y x$;
(iv) The following are equivalent:
(a) $\sum_{n=1}^{\infty} | < x, e_n > |^2 = \|x\|^2$ (Parseval's identity);
(b) $x = \sum_{n=1}^{\infty} < x, e_n > e_n$, where the series converges in the norm of X;
(c) $x \in Y$.

Proof. (i) Direct calculations.

(ii) Immediately follows from (i).

(iii) By proposition 7.1, series $\sum_{n=1}^{\infty} < x, e_n > e_n$ converges. Let us denote its sum by y. By definition of Y, $y \in Y$. Since $< x - y, e_j >= 0$ for any $j \in \mathbb{N}$, one cam claim $z = x - y \in Y^{\perp}$. So, $x = y + z$, where $y \in Y$ and $z \in Y^{\perp}$ and thus $P_Y x = y$.

(iv) (a)$\Longleftrightarrow$(b) follows from (i), (b)$\Longrightarrow$(c) is obvious, (c)$\Longrightarrow$(b), since $x - \sum_{n=1}^{\infty} < x, e_n > e_n \in Y^{\perp} \cap Y$. □

Given $x \in X$, the converging series $\sum_{n=1}^{\infty} < x, e_n > e_n$ is called the Fourier series of x.

Now, let us discuss several simple examples of orthonormal sequence in L_2-spaces.

1. Consider the space of all polynomials, and the sequence 1, t, t^2,... and use the Gram-Schmidt method to obtain orthonormal polynomials $p_n(t)$ of degree n. For the inner product $f, g >= \int_{-1}^{1} f\overline{g}dx$, the resulting orthonormal polynomials are, up to normalisation, the classical Legendre polynonials.

2. Let $X = L_2(0, \infty; e^{-t}dt)$, so the inner product given by

$$< f, g >= \int_0^\infty f(t)\overline{g}(t)e^{-t}dt,$$

the orthonormal polynomials obtained by applying the Gram-Schmidt method to 1, t, t^2,... are the Laguerre polynomials $L_n(t)$.

3. For the inner product

$$< f, g >= \int_{\mathbb{R}} f(t)\overline{g}(t)e^{-t^2}dt,$$

orthonormal polynomials are, up to normalisation, Hermite polynomial.

4. The last example is the Hilbert-Schmidt Theorem and its not easy.

Theorem 7.3. *Let K be a compact self-adjoint operator in a complex Hilbert space X. There exists an orthonormal sequence $\{e_n\}_{n=1}^N$, $N \leq \infty$, consisting of eigenvectors of K belonging to a non-zero eigenvalue λ_n of K with the following property. For any $x \in X$, we have a unique representaion*

$$x = \sum_{n=1}^N c_n e_n + x',$$

where $x' \in \operatorname{Ker} K$. If $N = \infty$, then $\lambda_k \to 0$ as $k \to \infty$.

Proof. As it follows from Theorem 6.13, the set of non-zero eigenvalues is countable. We list them in the following order

$$|\lambda_1| \geq |\lambda_2| \geq ... \geq |\lambda_n| \geq ....$$

Step 1 By Theorem 6.12, we know that

$$\|K\| = |\lambda_1| = \sup\{| < Kx, x > | : \ \|x\| = 1\}$$

and there exists an eigenvector e_1 such that $\|e_1\| = 1$ and $Ke_1 = \lambda_1 e_1$.

Step 2 Now, we argue by induction, assuming that there are eigenvectors e_1, e_2,...,e_n such that $\|e_j\| = 1$, $Ke_j = \lambda_j$, and $< e_j, e_m >= 0$ for $j \neq m$ and $j, m = 1, 2, ..., n$, see Proposition 6.5, (iii). Moreover,

$$|\lambda_j| = \sup\{| < Kx, x > | : \ \|x\| = 1, < x, e_m >= 0, \ m = 1, 2, ..., j-1\}$$

Consider a closed subspace $M_n = \text{span}\,\{e_1, e_2, ..., e_n\}$. Then $x = M_n \oplus M_n^\perp$. Obviously, K is invariant with respect M_n, i.e., $K(M_n) \subseteq M_n$. Let us show that $M_n^\perp$ is invariant with respect to K, i.e., $K(M_n^\perp) \subseteq M_n^\perp$, as well. Indeed, assume that there exists $y \in K(M_n^\perp) \setminus M_n^\perp$, then $y = m + m^\perp = Kx$, with $x \in M_n^\perp$, $m \in M_n$, and $m^\perp \in M_n^\perp$. Since M_n is invariant with respect to K, $\|m\|^2 =< Kx, m >=< x, Km >= 0$ and thus $y \in M_n^\perp$. If denote by $K_n : M_n^\perp \to M_n^\perp$ the restriction of K to M_n, we can repeat arguments of Step 1 replacing X with M_n and K with K_n and get $e_{n+1} \in M_n^\perp$ with $\|e_{n+1}\| = 1$ and $Ke_{n+1} = K_n e_{n+1} = \lambda_{n+1} e_{n=1}$ and

$$|\lambda_{n+1}| = \sup\{| < K_n x, x > | : \ x \in M_n^\perp, \ \|x\| = 1\} =$$

$$= \sup\{| < Kx, x > | : \ \|x\| = 1, < x, e_m >= 0, \ m = 1, 2, ..., n\}.$$

Step 3 Here, we should consider two case. In the first case, after finite steps, we get $< Kx, x >= 0$ for all $x \in M_{n_0}^\perp$ for some n_0. This implies that $M_{n_0}^\perp = \text{Ker}\, K$. Indeed, since K is self-adjoint, the identity $< K(x+y), x+y >= 0$ for any $x, y \in M_{n_0}^\perp$ implies $< Kx, y >= 0$ for the same x and y. Since K is invariant with respect $M_{n_0}^\perp$, one can let $y = Kx$ and get $Kx = 0$ for $x \in M_{n_0}^\perp$.

In the second case, $< Kx, x >$ is not identically equal to zero on $M_n^\perp$ for all n. In this case, $\lambda_n \to 0$. If not $\underline{|\lambda_n| \geq \delta \text{ for all } n.\text{ Contradiction}}$ follows from Theorem 6.13. Now, let $M = \overline{\text{span}\,\{e_1, e_2, ..., e_n, ...\}}$ and $X = M \oplus M^\perp$. Our aim is to show that $\text{Ker}\, K = M^\perp$. To this end, we first notice that K is invariant with respect to $M^\perp$. So, if $x \in \text{Ker}\, K$, then $0 =< Kx, e_n >=< x, Ke_n >= \overline{\lambda}_n < x, e_n >$ for any n, and thus $x \in M^\perp$ and $\text{Ker}\, K \subseteq M^\perp$. To show converse, we first notice that if $< Kx, x > \equiv 0$ on $M^\perp$, then the opposite inclusion is trivially true by repeating arguments of Step 2. Otherwise, we have

$$0 < |\lambda| = \sup\{| < Kx, x > | : \ \|x\| = 1, \ x \in M^\perp\} \leq |\lambda_n| \to 0.$$

Uniqueness is easy. □

7.2 Completeness

Definition 7.4. *If an orthonormal sequence* $\{e_n\}$ *in a Hilbert space in* X *satisfies* $X = \overline{\text{span}\{e_1, e_2, ..., e_n, ...\}}$*, it is said to be a complete orthonormal sequence or an orthonormal basis.*

It is worthy to note that if $\{e_j\}$ is an orthonormal basis in X, then

$$x = \sum_{j=1}^{\infty} < x, e_j > e_j$$

for any $x \in X$, see Proposition 7.2.

Consider examples of complete orthogonal sequences.

1. The standard orthonormal set in l_2 (easy to check).

2. The trigonometric orthonormal set $\{e_n : n \in \mathbb{Z}\}$ where $e_n(t) = (2\pi)^{-\frac{1}{2}} e^{int}$, where in $L_2(-\pi, \pi)$ (to be proved later).

3. The functions $(\frac{n+1}{\pi})^{\frac{1}{2}} z^n$ in $A^2(\mathbb{D})$ (see Problem sheet, Qn 25).

4.

Theorem 7.5. *Let* X *be an infinite-dimensional Hilbert space.* X *is separable if and only if it has a complete orthonormal sequence.*

Proof. Let X be separable and a set $\{x_n\}$ be dense in it. Without loss of generality we may assume that x_1, x_2,...,x_n are linear independent for any n. Then we can find a orthonormal sequence $\{e_n\}$ using Gram-Schmidt method. So, $x_n \in \text{span}\{e_1, e_2, ..., e_n\}$. This implies $\{e_n\}$ is an orthonormal basis in X. Converse is obvious. □

5. Now, we can give another version of Hilbert-Schmidt Theorem which is very important in applications.

Theorem 7.6. *Let* K *be a compact self-adjoint operator in a separable Hilbert space* X*. There exists an orthonormal basis of* X *consisting of eigenvalues of the operator* K*.*

Proof. Our first remark is that $\text{Ker}\, K$ is a separable Hilbert space itself. By Theorem 7.5, there exists a orthonormal basis $\{e'_n\}$ of $\text{Ker}\, K$ which consists of eigenvectors belonging zero eigenvalue. Now, it remains to use Theorem 7.3. According to it, there exists an orthonormal sequence $\{e''_n\}$ consisting of eigenvectors belonging to non-zero eigenvalues such that $\{e'_n\} \cup \{e''_n\}$ is orthonormal basis in X. □

8 Classical Fourier Series

Given $f \in L_1(-\pi, \pi)$, consider the classical Fourier series of f

$$\sum_{k \in \mathbb{Z}} f_k e^{ikx} \tag{8.1}$$

for $-\pi < x < \pi$ with $f_k = \int\limits_{-\pi}^{\pi} f(x) e^{-ikx} dx$.

Since the system $\{e^{ikx}\}_{k \in \mathbb{Z}}$ is orthogonal, series (8.1) can be re-written in the form

$$\sum_{k \in \mathbb{Z}} < f, e_k > e_k, \tag{8.2}$$

where $e_k(x) = \frac{1}{\sqrt{2\pi}} e^{-ikx}$ and $< f, g >= \int\limits_{-\pi}^{\pi} f \overline{g} dx$. We can hope that this series converges in $L_2(-\pi, \pi)$.

Assume that f is 2π-periodic function on $\mathbb{R}$ and $f \in L_1(-\pi, \pi)$. Let us calculate the n^{th} partial sum of the Fourier series of a fixed function f is given by

$$s_n(x) = \sum_{j=-n}^{n} f_j e^{ijx} =$$

$$= \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} f(t) \sum_{j=-n}^{n} e^{ij(x-t)} dt =$$

$$= \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} f(u) \frac{\sin((n+\frac{1}{2})(x-u))}{\sin((x-u)/2)} du =$$

$$= \frac{1}{\pi} \int\limits_{0}^{\pi} \frac{(f(x+t) + f(x-t))}{2} \frac{\sin(n+\frac{1}{2})t}{\sin t/2} dt.$$

By the Riemann-Lebesgue Lemma, part A, see also Qn 8 and Qn 19 for an independent proof,

$$\frac{1}{\pi} \int\limits_{\delta}^{\pi} \frac{(f(x+t) + f(x-t))}{2} \frac{\sin(n+\frac{1}{2})t}{\sin t/2} dt \to 0$$

as $n \to \infty$ for any positive δ. Therefore, the classical Fourier series (8.1) converges at point $x \in]-\pi, \pi[$, if and only if, for some $\delta > 0$, there exists

$$\lim_{n\to\infty} \frac{1}{\pi} \int_0^\delta \frac{(f(x+t)+f(x-t))}{2} \frac{\sin(n+\frac{1}{2})t}{\sin t/2} dt.$$

Moreover, the sum of the series is equal to the above limit.

Unfortunately, we need to put more assumptions on f if we wish convergence of the series at least a.e. in $]-\pi, \pi[$. It follows from the celebrated Kolmogorov counter-example of $f \in L_1(-\pi, \pi)$ whose the classical Fourier series diverges a.e. in $]-\pi, \pi[$.

Assume that our function f is Hölder continuous at point x, i.e. there exist numbers $A > 0$, $0 < \alpha \le 1$ and $\delta_0 > 0$ such that

$$|f(x+z) - f(x)| \le A|z|^\alpha, \quad |z| \le \delta_0. \tag{8.3}$$

Lemma 8.1. *Let f be 2π-periodic and $f \in L_1(-\pi, \pi)$. Assume that condition (8.3) holds at a point $-\pi < x < \pi$. Then*

$$\lim_{n\to\infty} s_n(x) = f(x).$$

Proof. We have

$$|s_n(x) - f(x)| = \Big|\frac{1}{\pi} \int_0^\pi \frac{(f(x+t)+f(x-t))}{2} \frac{\sin(n+\frac{1}{2})t}{\sin t/2} dt - f(x)\Big| \le I_1 + I_2,$$

where

$$I_1 = \Big|\frac{1}{\pi} \int_\delta^\pi \frac{(f(x+t)+f(x-t))}{2} \frac{\sin(n+\frac{1}{2})t}{\sin t/2} dt\Big| \to 0$$

for any $\delta > 0$ as $n \to \infty$ (by the Riemann-Lebesgue Lemma) and

$$I_2 = \Big|\frac{1}{\pi} \int_0^\delta \frac{(f(x+t)+f(x-t))}{2} \frac{\sin(n+\frac{1}{2})t}{\sin t/2} dt - f(x)\Big| \le J_1 + J_2,$$

where

$$J_1 = \Big|\frac{1}{\pi} \int_0^\delta \frac{(f(x+t)+f(x-t)-2f(x))}{2} \frac{\sin(n+\frac{1}{2})t}{\sin t/2} dt\Big|$$

and

$$J_2 = |f(x)|\Big|\frac{1}{\pi}\int\limits_0^\delta \frac{\sin(n+\frac{1}{2})t}{\sin t/2}dt - 1\Big|.$$

Let us show that $J_2 \to 0$ as $n \to \infty$. Indeed,

$$\frac{1}{\pi}\int\limits_0^\delta \frac{\sin(n+\frac{1}{2})t}{\sin t/2}dt - 1 =$$

$$= \frac{2}{\pi}\int\limits_0^\delta \frac{\sin(n+\frac{1}{2})t}{t}dt - \frac{1}{\pi}\int\limits_0^\delta \sin((n+\frac{1}{2})t)\Big(\frac{1}{t/2} - \frac{1}{\sin t/2}\Big)dt = K_1 + K_2.$$

Since the function $\frac{1}{t/2} - \frac{1}{\sin t/2}$ is bounded in the interval $]0, \delta[$, $K_2 \to 0$ as $n \to \infty$. In the integral K_1, we make the change of variables $s = (n + 1/2)t$ and find

$$K_1 = \frac{2}{\pi}\int\limits_0^{(n+1/2)\delta} \frac{\sin s}{s}ds \to 1$$

as $n \to \infty$.

For J_1, we have, assuming that $\delta \le \delta_0$,

$$J_1 \le \frac{A}{\pi}\int\limits_0^\delta \frac{t^\alpha}{\sin t/2}dt.$$

Since $t/\pi \le \sin t/2$ if $0 \le t \le \pi$, $J_1 \le \frac{A}{\alpha}\delta^\alpha$. Passing to the limit as $n \to \infty$, we find

$$\limsup_{n\to\infty} |s_n(x) - f(x)| \le \frac{A}{\alpha}\delta^\alpha.$$

After taking the limit as $\delta \to 0$, we complete the of the lemma. □

Theorem 8.2. *The trigonometric sequence $\{e_k\}_{k\in\mathbb{Z}}$ is orthonormal basis of $L_2(-\pi, \pi)$.*

Proof. We going to use standard density arguments. Consider a 2π-periodic function $f \in \mathcal{X} = L_2(-\pi, \pi)$ (and of course belonging to $L_1(-\pi, \pi)$ as well) that is differentiable at any point $s \in]-\pi, \pi[$. We have proved $f(x) = \sum\limits_{k\in\mathbb{Z}} <$

$f, e_k > e_k(x)$ for any $x \in]-\pi, \pi[$. On the other hand, by Proposition 6.5,(iii), the series converges to $P_{\mathcal{Y}} f$ in $L_2(-\pi, \pi)$, where $\mathcal{Y} = \overline{\text{span}\,\{e_n\}}$. Clearly, this series converges a.e. in $]-\pi, \pi[$ and thus $P_{\mathcal{Y}} f = f$, i.e., $f \in \mathcal{Y}$. Our theorem will be proven if show that $\mathcal{Y}$ is dense.

Now, let $f \in L_2(-\pi, \pi)$ be arbitrary. Let $\tilde{f}$ be a 2π-periodic extension f to the whole $\mathbb{R}$. For example $\tilde{f}(x) = f(x)$ if $x \in]-\pi, \pi]$ and $\tilde{f}(x) = f(x - 2\pi k)$ if $x \in]2\pi k - \pi, 2\pi k + \pi]$, $k \in \mathbb{Z}$. Let us take a cut-off bounded smooth function $\varphi \geq 0$ such $\varphi(x) = 1$ if $|x| \leq 5\pi$ and $\varphi(x) = 0$ if $|x| \geq 10\pi$. We then let $h = \tilde{f}\varphi$ in $\mathbb{R}$. Obviously, $h \in L_2(\mathbb{R})$.

In our further arguments, we are going to use mollification described in Subsection 4.4, see example 3. Take a differentiable non-negative function g which is equal to zero if $|x| > 1$ and $\int\limits_{\mathbb{R}} g dx = 1$, let $g_n(t) = ng(tn)$ and consider $h_n = h * g_n$. It has been shown that $h_n \to h$ in $L_2(\mathbb{R})$. Moreover, functions h_n are differentiable at each point of $\mathbb{R}$. It remains to consider $\tilde{h}_n(x)$, see definition of $\tilde{f}$. As it has been proven, $\tilde{h}_n \in \mathcal{Y}$ and since

$$\|f - \tilde{h}_n\|_{L_2(-\pi,\pi)} = \|\tilde{f} - \tilde{h}_n\|_{L_2(-\pi,\pi)} = \|\tilde{f}\varphi - \tilde{h}_n\|_{L_2(-\pi,\pi)} = \|h - \tilde{h}_n\|_{L_2(-\pi,\pi)}$$

$$= \|h - h_n\|_{L_2(-\pi,\pi)} \leq \|h - h_n\|_{L_2(\mathbb{R})} \to 0$$

as $n \to \infty$. □