B3.4 Algebraic Number Theory, Hilary 2020

Exercises 2

Q 1. Suppose that α is an algebraic integer of degree n, with monic minimal polynomial $m_\alpha \in \mathbb{Z}[X]$. Let $K = \mathbb{Q}(\alpha)$. Show that

$$\mathrm{disc}_{K/\mathbb{Q}}(1, \alpha, \dots, \alpha^{n-1}) = (-1)^{n(n-1)/2}\mathbf{N}_{K/\mathbb{Q}}(m'_\alpha(\alpha)).$$

Using this, compute $\mathrm{disc}_{K/\mathbb{Q}}(1, \alpha, \alpha^2)$, where $K = \mathbb{Q}(\alpha)$ with $\alpha = 2^{1/3}$.

The next five questions are related and discuss the cyclotomic field $K = \mathbb{Q}(\zeta_p)$, where $\zeta_p := e^{2\pi i/p}$ and p is an odd prime.

Q 2. Show that the degree $[K : \mathbb{Q}]$ is $p - 1$.

Q 3. Evaluate $\mathbf{N}_{K/\mathbb{Q}}(1 - \zeta)$.

Q 4. Show that $\frac{1}{p}(\zeta - 1)^{p-1}$ is an algebraic integer.

Q 5. Evaluate $\mathrm{disc}_{K/\mathbb{Q}}(1, \zeta, \dots, \zeta^{p-2})$. (*Hint: you may want to use Question 1 and the answer to Question 3.*)

Q 6. (i) Suppose that $c_0, c_1, \dots, c_{p-2}$ are integers and that

$$\frac{1}{p}(c_0 + c_1(\zeta - 1) + \cdots + c_{p-2}(\zeta - 1)^{p-2}) \in \mathcal{O}_K.$$

Show that all the c_i are divisible by p. (*Hint: suppose not, and let r be minimal such that $p \nmid c_r$. You may wish to recall Questions 3 and 4.*)

(ii) Show that $1, \zeta, \dots, \zeta^{p-2}$ is an integral basis for $\mathcal{O}_K$.

Q 7. Let K be a number field. We say that K is *norm-Euclidean* if $\mathcal{O}_K$ is a Euclidean domain with respect to the norm function: that is, given $a, b \in \mathcal{O}_K\setminus\{0\}$ we may find $q, r \in \mathcal{O}_K$ such that $a = qb+r$ with $|\mathbf{N}_{K/\mathbb{Q}}(r)| < |\mathbf{N}_{K/\mathbb{Q}}(b)|$.

(i) Show that a norm Euclidean domain is a principal ideal domain.

(ii) Let $K = \mathbb{Q}(\sqrt{-7})$. Show that K is norm-Euclidean.

Q 8. Let $K = \mathbb{Q}(\sqrt{-7})$. In this question you may assume (as follows from Question 7) that $\mathcal{O}_K$ is a PID.

(i) Factor 2 and $\sqrt{-7}$ into irreducibles in $\mathcal{O}_K$.

(ii) Suppose that $7 \nmid x$. Show that $2x + \sqrt{-7}$ and $2x - \sqrt{-7}$ are coprime.

(iii) Show that there are no integer solutions to the equation $4x^2 + 7 = y^3$.

Q 9. Let $K = \mathbb{Q}(\sqrt{-p})$, where p is a prime congruent to $1(\mathrm{mod}\, 4)$. Show that $\mathcal{O}_K$ is not a principal ideal domain.

ben.green@maths.ox.ac.uk