B3.4 Algebraic Number Theory, Hilary 2020

Exercises 3

Q 1. Factor the ideal $\mathfrak{a}$ into prime ideals in $\mathcal{O}_K$ in the following cases.

(i) $\mathfrak{a} = (5)$, $K = \mathbb{Q}(\sqrt{-6})$;

(ii) $\mathfrak{a} = (6, 1 + \sqrt{-17})$, $K = \mathbb{Q}(\sqrt{-17})$;

(iii) $\mathfrak{a} = (2 + 3\sqrt{10})$, $K = \mathbb{Q}(\sqrt{10})$;

(iv) $\mathfrak{a} = (5 + 2^{2/3})$, $K = \mathbb{Q}(2^{1/3})$ (you may use the fact, proven on Sheet 1, that $\mathcal{O}_K = \mathbb{Z}[2^{1/3}]$).

Q 2. Let $K = \mathbb{Q}(\sqrt{10})$. Show that $\mathcal{O}_K$ is not a principal ideal domain.

Q 3. Suppose that $K = \mathbb{Q}(\sqrt{d})$, d squarefree and $d \neq 0, 1$. Show that whether (2) ramifies, is inert, or splits completely on $\mathcal{O}_K$ depends only on the value of $d (\mathrm{mod}\, 8)$, and classify the different possibilities.

Q 4. Let $K = \mathbb{Q}(\sqrt{15})$. Show that there is a unique ideal in $\mathcal{O}_K$ of norm 12. Is it principal? Find all ideals containing 8.

Q 5. Let $\mathcal{O}_K$ be the ring of integers of a number field, and let p be a rational prime. Show that p ramifies in $\mathcal{O}_K$ if and only if the ring $\mathcal{O}_K/(p)$ has a nilpotent element, that is to say a nonzero element x for which $x^n = 0$ for some n.

Q 6. Let $K = \mathbb{Q}(\sqrt{-210})$. Show that $h_K \geqslant 8$.

Q 7. Let $f(X) = X^3 - X^2 - 2X - 8$. Recall from Sheet 1 that f is irreducible, and that if α is a root of f and $K = \mathbb{Q}(\alpha)$, then e_1, e_2, e_3 is an integral basis for $\mathcal{O}_K$, where $e_1 = 1$, $e_2 = \alpha$ and $e_3 = \frac{1}{2}\alpha(\alpha + 1)$.

(i) Show that the linear maps $\psi_v : \mathcal{O}_K \to \mathbb{F}_2$ defined by $\psi_v(e_i) = v_i$ are ring homomorphisms, for the following values of v: (1) $v = (1, 0, 0)$; (2) $v = (1, 1, 0)$; (3) $v = (1, 0, 1)$. Conclude that $\mathcal{O}_K/(2) \cong \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2$.

(ii) Explain why it follows from (i) that (2) splits completely in $\mathcal{O}_K$. *Hint: recall Question 5.*

(iii) Show that $\mathcal{O}_K$ does not have a power integral basis.

Q 8. Let $\mathcal{O}_K$ be the ring of integers of a number field.

(i) Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_K$ and n a positive integer. Explain why every proper ideal of the quotient ring $\mathcal{O}_K/\mathfrak{p}^n$ is of the form $\mathfrak{p}^i/\mathfrak{p}^n$ for some i.

(ii) Show that $\mathfrak{p}^i/\mathfrak{p}^n$ is principal. (*Hint: first try the case* $n = i + 1$)

(iii) Let $\mathfrak{a}$ be an arbitrary ideal in $\mathcal{O}_K$. Show that every ideal in $\mathcal{O}_K/\mathfrak{a}$ is principal. (*Hint: Chinese Remainder Theorem.*)

(iv) Conclude that every ideal $\mathfrak{a}$ in $\mathcal{O}_K$ is generated by at most two elements.

Q 9. In this question, do not worry about questions of convergence. If K is a number field, the *Dedekind ζ-function* is defined by

$$\zeta_K(s) := \sum_{\mathfrak{a} \subseteq \mathcal{O}_K, \mathfrak{a} \neq 0} (N\mathfrak{a})^{-s}.$$

Explain why

$$\zeta_K(s) = \prod_{\mathfrak{p}} (1 - N(\mathfrak{p})^{-s})^{-1},$$

where the product is over prime ideals in $\mathcal{O}_K$. Hence, show that

$$\zeta_{\mathbb{Q}(i)}(s) = \zeta_{\mathbb{Q}}(s) L(s),$$

where

$$L(s) = 1 - 3^{-s} + 5^{-s} - 7^{-s} + \dots$$

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