B3.4 Algebraic Number Theory, Hilary 2020

Exercises 4

Question 1. Find all quadratic fields for which the Minkowski bound is strictly less than 2. What is the class number of these fields?

Question 2. Show that $\mathrm{Cl}(K)$ is cyclic of order two, where $K = \mathbb{Q}(\sqrt{-37})$.

Question 3. Find $\mathrm{Cl}(K)$, where $K = \mathbb{Q}(\sqrt{-6})$.

Question 4. Let K be a number field, other than $\mathbb{Q}$. Show that $\Delta_K > 1$.

Question 5. Find $\mathrm{Cl}(K)$, where $K = \mathbb{Q}(\sqrt{-34})$.

Question 6. Find $\mathrm{Cl}(K)$, where $K = \mathbb{Q}(\sqrt{65})$.

Question 7. Find all integer solutions to the equation $y^2 + 74 = x^3$. (You may assume that $h_{\mathbb{Q}(\sqrt{-74})} = 10$.)

Question 8. Show that the ring of integers in $\mathbb{Q}(2^{1/3})$ is a principal ideal domain (any results about this field established on previous sheets may be used without proof).

Question 9. (i) Let Λ be a lattice in $\mathbb{R}^2$ which has no nonzero vector x with $\|x\| \leqslant 1$. Show that $\det(\Lambda) \geqslant \frac{1}{2}\sqrt{3}$.

 (ii) Deduce a (small) improvement to the Minkowski bound for imaginary quadratic fields.

 (iii) Briefly comment on the implications for the Rabinowicz phenomenon.

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