**Professor Joyce** **B3.3 Algebraic Curves** **Hilary Term 2019**

**Initial problem sheet**

*Here is an initial sheet of problems on B3.3 Algebraic Curves. Starred problems are harder. The sheet is entirely voluntary, there will be no classes on it, and you should not hand answers in or ask people to mark them. Model answers are on the Institute web site.*

**1.** Let $[2,1],[1,1],[3,4]$ be points in the projective line $\mathbb{CP}^1$. Find representative vectors $v_1,v_2,v_3$ for these points which satisfy $v_1+v_2+v_3=0$.

**2.** Explain why two photographs taken from the same point, but with the camera pointed in different directions, are related by a projective transformation.

A photograph shows four fence posts beside a straight road. On the photograph, the distances between successive fence posts are 4 inches, 3 inches and 2 inches. Is it possible that the fence posts are evenly spaced? Give reasons.

**3.** If a line with slope $t$ intersects the circle $x^2+y^2=1$ in the points $(-1,0)$ and $(x,y)$, show that $x$ and $y$ are both rational functions of $t$. (A rational function is one that can be written as the quotient of two polynomials.) By taking $t=p/q$ to be a rational number, construct the general solution of the equation $x^2+y^2=z^2$ for which $x,y,z$ are coprime integers.

**4**$^*$. Suppose $p(t),q(t)$ and $r(t)$ are pairwise coprime, complex polynomials in $t$ satisfying $p(t)^3+q(t)^3+r(t)^3\equiv 0$. Let $\omega=e^{2\pi i/3}$, so that $\omega^3=1$. Then the equation $p(t)^3+q(t)^3+r(t)^3=0$ may be rewritten as

$$\bigl(p(t)+q(t)\bigr)\bigl(\omega\,p(t)+\omega^2q(t)\bigr)\bigl(\omega^2p(t)+\omega\,q(t)\bigr)=\bigl(-r(t)\bigr)^3.$$

**(i)** Show that $p(t)+q(t)$, $\omega\,p(t)+\omega^2q(t)$ and $\omega^2p(t)+\omega\,q(t)$ are pairwise coprime.

**(ii)** Show that there exist pairwise coprime, complex polynomials $\alpha(t),\beta(t)$ and $\gamma(t)$, such that $p(t)+q(t)\equiv\alpha(t)^3$, $\omega\,p(t)+\omega^2q(t)\equiv\beta(t)^3$, and $\omega^2p(t)+\omega\,q(t)\equiv\gamma(t)^3$.

**(iii)** Deduce that $\alpha(t)^3+\beta(t)^3+\gamma(t)^3\equiv 0$.

[*Two polynomials are coprime if they have no nontrivial common factor.*]

**5***. Using your answer to Question 4, prove that $p(t)$, $q(t)$ and $r(t)$ must be constant.

[*Hint: consider the degrees of* $p, q, r$ *and* $\alpha, \beta, \gamma$.]

**6***. Using Questions 4 and 5, show that there do not exist nonconstant rational functions $x(t)$, $y(t)$, such that $x(t)^3 + y(t)^3 + 1 \equiv 0$. How does this compare with Question 3?

DDJ