Professor Joyce **B3.3 Algebraic Curves** **Hilary Term 2019**

Problem Sheet 1

1(a) Let $\mathbb{Z}_2$ be the field with two elements $\{0,1\}$. Show that the number of points in n-dimensional projective space over $\mathbb{Z}_2$ is $2^{n+1}-1$. How many projective lines are there in this space?

(b) What are the answers if $\mathbb{Z}_2$ is replaced by $\mathbb{Z}_p$, the field with p elements for p prime?

2. Find a 1-1 correspondence between the lines in $\mathbb{R}^2$ and the complement of a point in $\mathbb{P}(\mathbb{R}^3)$.

3. Let $\mathbb{P}(V)$ be a projective space of dimension 3, and let L_1, L_2, L_3 be non-intersecting projective lines in $\mathbb{P}(V)$.

(a) Show that there are unique isomorphisms $\alpha : L_1 \to L_2$, $\beta : L_1 \to L_3$ such that $x, \alpha(x), \beta(x)$ are collinear for each $x \in L_1$.

(b) Suppose $\phi : L_1 \to L_1$ is a projective transformation of L_1. Show that there exists a unique projective transformation $\Phi : \mathbb{P}(V) \to \mathbb{P}(V)$ such that $\Phi(L_j) = L_j$ and $\Phi|_{L_1} = \phi$.

4(a) Define $\Phi : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^3$ by

$$\Phi([x_0,x_1],[y_0,y_1]) = [x_0y_0, x_0y_1, x_1y_0, x_1y_1].$$

Show that Φ is well defined and injective, and the image of Φ is of the form $\{[z_0,z_1,z_2,z_3] \in \mathbb{CP}^3 : Q(z_0,\ldots,z_3) = 0\}$ for a homogeneous quadratic polynomial Q.

(b) Write down a similar map $\Psi : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^3$ with image

$$\{[z_0,z_1,z_2,z_3] \in \mathbb{CP}^3 : z_0^2 + \cdots + z_3^2 = 0\}.$$

5. Find explicitly the projective transformation of $\mathbb{CP}^2$ that takes the points $[1,0,0]$, $[1,1,0]$, $[0,0,1]$ and $[0,1,1]$ to $[1,2,3]$, $[1,0,1]$, $[0,1,0]$ and $[1,0,-1]$ respectively.

6. (Optional). Find a projective transformation of $\mathbb{CP}^2$ that takes the lines $x+2y+3z=0$, $x+z=0$, $y=0$ and $x-z=0$ to the lines $x=0$, $x+y=0$, $z=0$ and $y+z=0$ respectively. Compare this question to q. 5. What does this tell you about the relation between lines and points in $\mathbb{CP}^2$?

DDJ