Professor Joyce **B3.3 Algebraic Curves** **Hilary Term 2019**

Problem Sheet 3

1. Let $P(x, y, z)$ be a homogeneous polynomial of degree d defining a non-singular curve C in $\mathbb{CP}^2$.

(i) Write down Euler's relation for P, P_x, P_y, P_z. Deduce that the Hessian determinant satisfies:

$$z\mathcal{H}_P(x, y, z) = (d-1)\det\begin{pmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_x & P_y & P_z \end{pmatrix}.$$

(ii) Deduce further that:

$$z^2\mathcal{H}_P(x, y, z) = (d-1)^2\det\begin{pmatrix} P_{xx} & P_{xy} & P_x \\ P_{yx} & P_{yy} & P_y \\ P_x & P_y & dP/(d-1) \end{pmatrix}.$$

(iii) Deduce that if $P(x, y, 1) = y - g(x)$ then $[a, b, 1]$ is a point of inflection of C if and only if $b = g(a)$ and $g''(a) = 0$.

This shows the lectures definition of points of inflection corresponds to the usual notion of a point of inflection of the graph of a function $g(x)$ on $\mathbb{R}$ or $\mathbb{C}$.

2. By a projective transformation, show how to take the cubic $x^3+y^3+z^3 = 0$ into the form $y^2z = x(x-z)(x-\lambda z)$, for λ which you should determine.

3(i) Show that given any five points in $\mathbb{CP}^2$ there is at least one conic passing through them.

(ii) Let C be a quartic curve (i.e. of degree 4), with four singular points. By choosing an appropriate conic D and using the strong form of Bézout's theorem (involving intersection multiplicities $I_p(C, D)$) prove that C must be reducible.

(iii) Show that $y^4 - 4xzy^2 - xz(z-x)^2 = 0$ is a quartic with three singular points.

4(i) Let U be a connected open subset of $\mathbb{C}$, and let $f : U \to \mathbb{C}$ be holomorphic. Show that if $a \in U$, then for sufficiently small real positive r, we have:

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})\, \mathrm{d}\theta.$$

(ii) Deduce that if $|f|$ has a local maximum at $a \in U$, then $|f|$ is constant on some neighbourhood of a.

(iii) Deduce that if $|f|$ has a local maximum at $a \in U$, then f is constant on U.

(iv) Now suppose S is a connected, compact Riemann surface and $f : S \to \mathbb{C}$ is a holomorphic function. Show that f is constant.

5. Compute the resultant (with respect to x) of the two polynomials $x^2 - y^2 + xz$ and $-x^2 - y^2 + xz$ over $\mathbb{C}$. Hence compute the intersection multiplicity of the curves defined by these polynomials at the point $p = [0, 0, 1]$ in $\mathbb{CP}^2$.

6. (Optional). Find the points of intersection and the intersection multiplicity for the projective curves defined by $z^6 + y^2z^4 + x^6$ and $z^4 + y^2z^2 - x^4$ over $\mathbb{C}$. [*Hint:* instead of launching into brute-force calculations, try to use the axiomatic properties of intersection multiplicities, as in Hitchin notes Theorem 16, or Kirwan Theorem 3.18.]

DDJ