B2.2 Commutative Algebra
Problem Sheet 1

R denotes a commutative ring with 1. We assume $1 \neq 0$. 'Module' will mean 'R-module'.

1. Let Y be a multiplicatively closed subset of a ring R, with $0 \notin Y$. Prove that any ideal P of R maximal with respect to $P \cap Y = \emptyset$ is prime.

2. Let $P_1, \ldots, P_n$ be prime ideals in a ring, and I any ideal. Show that if $I \subseteq P_1 \cup P_2 \cup \cdots \cup P_n$ then $I \subseteq P_j$ for some j.

3. Let $\theta : M \to N$ and $\phi : N \to M$ be module homomorphisms such that $\theta(\phi(x)) = x$ for all $x \in N$. Show that $M = \ker\theta \oplus \phi(N)$.

4. Let M be an extension of A by B (modules), i.e. $A \leqslant M$ and $M/A \cong B$.

   (i) Show that if M is finitely generated then so is B.

   (ii) Show that if both A and B are finitely generated then so is M.
   Is the converse true, in general?

   (iii) Deduce: M is Noetherian if and only if both A and B are Noetherian.

5. Let F be a free module with basis X. Let B be an arbitrary module and let $\theta : X \to B$ be an arbitrary mapping. Show that there exists a unique module homomorphism $\theta^* : F \to B$ such that $\theta^*(x) = \theta(x)$ for every $x \in X$. Deduce:

   (i) every module is a homomorphic image of a free module, and an n-generator module is a homomorphic image of R^n;

   (ii) if R is Noetherian then every finitely generated R-module is Noetherian.

6. Let F be a free module and let $f : M \to F$ be an *epimorphism*, i.e. a surjective homomorphism. Show that there exists a homomorphism $h : F \to M$ such that $f \circ h = \mathrm{Id}_F$.

   Hint: for each $x \in X$, a basis for F, choose a pre-image of x under f.

   Deduce: if $N \leqslant M$ and M/N is free, then N has a complement in M, i.e. there exists a submodule C of M such that $M = N \oplus C$.

7. (i) Let M be a module. Show that if M is not Noetherian then M has a submodule N such that N is not finitely generated, but A *is* finitely generated whenever A is a submodule of M strictly containing N.

   (ii) Prove that if every prime ideal of the ring R is finitely generated then R is Noetherian.
   Hint: Take $M = R$ in (i). Then $N \geq BC$ where $N < B$ and $N < C$. Note that B/BC is a Noetherian R/C-module.