B2.2 Commutative Algebra
Problem Sheet 2

1. Let F be a field and let Y be a set of polynomials in k variables over F. Prove that there exist finitely many polynomials $f_1, \ldots, f_m \in Y$ such that for $u_1, \ldots, u_k \in F^k$,
$$f(u_1, \ldots, u_k) = 0 \quad \text{for all} \quad f \in Y \quad \Leftrightarrow \quad f_j(u_1, \ldots, u_k) = 0 \quad \text{for all} \quad j = 1, \ldots, m.$$

2. Let F be a field and let W be an algebraic set in F^k. Put $I := \mathcal{I}(W)$. Then
   (i) $W = \mathcal{V}(I)$.
   (ii) $I = \mathrm{rad}(I)$.
   (iii) Both F^k and $\emptyset$ are algebraic sets.
   (iv) The union of two algebraic sets in F^k is an algebraic set.
   (v) The intersection of any collection of algebraic sets in F^k is an algebraic set.

   Remark: (iii)-(v) say that the set of all algebraic sets in F^k is the set of closed sets in a certain topology: this is called the *Zariski topology.*

3. An algebraic set W is *irreducible* if it is not the union of two proper algebraic subsets.
   (i) Prove that W is irreducible if and only if $\mathcal{I}(W)$ is a prime ideal.
   (ii) Prove that every algebraic set is the union of finitely many irreducible algebraic subsets.
   Hint: consider a counterexample W with $\mathcal{I}(W)$ as big as possible.
   (iii) Let W be an algebraic set. What is wrong with the following argument? If $I = \mathcal{I}(W)$ then $I = \mathrm{rad}(I) = P_1 \cap \cdots \cap P_m$ with the P_i minimal primes of I. Then $W = \cup V_i$ where $V_i = \mathcal{V}(P_i)$ is irreducible.

4. Let F be a field and $R = F[t_1, \ldots, t_k]$. For $\mathbf{u} = (u_1, \ldots, u_k) \in F^k$ define $\epsilon_{\mathbf{u}} : R \to F$ by $e_{\mathbf{u}}(f) = f(\mathbf{u}) = f(u_1, \ldots, u_k)$ and set $\mu(\mathbf{u}) := \langle t_1 - u_1, t_2 - u_2, \ldots, t_k - u_k \rangle$, an ideal of R.
   (i) Prove that $R = \mu(\mathbf{u}) \oplus F$.
   (ii) Deduce that $\mu(\mathbf{u})$ is a maximal ideal in R.
   (iii) Prove $\mu(\mathbf{u}) = \ker e_{\mathbf{u}}$.
   (iv) Show that $\mu(\mathbf{u}) = \mu(\mathbf{v})$ if and only if $\mathbf{u} = \mathbf{v}$.
   (v) Prove that an ideal I of R is of the form $\mu(\mathbf{u})$ for some $\mathbf{u} \in F^k$ if and only if it has codimension one, i.e. $\dim_F(R/I) = 1$.

5. (i) Let F be a field and $R = F[t_1, \ldots, t_k]$. Let Y be a subset of R; then $\mu(\mathcal{V}(Y))$ is a set of maximal ideals of R: identify this set. (Now you have transformed geometry into algebra!)
   (ii) Suppose that F is *not* algebraically closed. Prove that not every maximal ideal of R is of the form $\mu(\mathbf{u})$.

6. Let F be an algebraically closed field and $f_1, \ldots, f_n$ polynomials in k variables over F. The system of simultaneous equations
$$\mathcal{F}: \quad f_1(x_1, \ldots, x_k) = 0, \ldots, f_n(x_1, \ldots, x_k) = 0$$
is said to be *inconsistent* if there exist polynomials $g_1, \ldots, g_n$ such that $f_1 g_1 + \cdots + f_n g_n = 1$.
   (i) Prove that the system of equations $\mathcal{F}$ has a solution in F^k if and only if it is not inconsistent.
   (ii) Suppose that all the f_i have coefficients in $\mathbb{Q}$, and that the system $\mathcal{F}$ has a solution in $\mathbb{C}^k$. Prove that it has a solution $(x_1, \ldots, x_k)$ with each x_i an algebraic number (i.e. algebraic over $\mathbb{Q}$).