B2.2 Commutative Algebra
Problem Sheet 3

R denotes a commutative ring with 1.

1. Let M be a finitely generated R-module and let $J = J(R)$ be the Jacobson radical of R. Prove that if $M = JM$ then $M = 0$. Is this true if M is not finitely generated?

2. (i) Prove that an integral extension of a Jacobson ring is Jacobson.

   (ii) Prove that the polynomial ring $F[t]$ over a field F is Jacobson.

   (iii) Find a principal ideal domain R such that $J(R) \neq 0$. Can you complete the sentence: A principal ideal domain is a Jacobson ring if and only if ... ?

3. Show that the maps c and e from Proposition 6.8 respect inclusions and finite intersections of ideals, and that e respects sums. Does the map c respect sums?

4. (i) Let $f(t_1, \ldots, t_n)$ be a polynomial over a field F. Suppose there exist infinite subsets $X_1, \ldots, X_n$ of F such that $f(x_1, \ldots, x_n) = 0$ for all $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$. Prove that f is the zero polynomial.

   (ii) Let U be an algebraic set in F^n and V an algebraic set in F^m . Show that $U \times V$ is an algebraic set in F^{n+m}.

   (iii) Let U be an algebraic set in F^n. A subset X of U is *dense* in U if U is the smallest algebraic set that contains X. Show that if X is a dense subset of U and Y is a dense subset of V then $X \times Y$ is dense in $U \times V$. How does this relate to (i)?

5. (i) Let M be a finitely generated R-module and $\phi : M \to M$ a module endomorphism. Prove that if ϕ is surjective then it is an isomorphism.
   Hint: Consider M as an $R[t]$-module where t acts as ϕ.

   (ii) Let $F = R^d$ be a free module and Y a generating set for F. Prove that if $|Y| \leqslant d$ then Y is a basis and $|Y| = d$.

6. A module M is said to have *length* $\lambda(M) = n$ if there is a chain of submodules

$$0 = M_0 < M_1 < \cdots < M_n = M$$

   and n is maximal; we say $\lambda(M) = \infty$ if there is no such maximal integer n.

   (i) Prove that length is additive on extensions of modules, i.e. if N is a submodule of M then $\lambda(M) = \lambda(N) + \lambda(M/N)$.

   (ii) Suppose that M is a Noetherian R-module and that $P^k M = 0$ for some maximal ideal P of R and some integer k. Show that M has finite length.