B2.2 Commutative Algebra
Problem Sheet 4

- An integral domain R is said to be *integrally closed* if R is its own integral closure in its field of fractions.
- If P is an ideal of R, the *height* $h(P)$ of P is the maximal length n of a chain of prime ideals

$$P_0 < P_1 < \cdots < P_n = P.$$

Note that $\dim(R)$ is the supremum of $h(P)$ over all prime ideals P (or maximal ideals, of course), of R.

1. Re-prove the 'Weak Nullstellensatz', Theorem 4.4:

   if E is a finitely generated F-algebra where $E \supseteq F$ are fields, then $[E : F]$ is finite

   from the 'Noether Normalisation Lemma', Theorem 8.8.

2. (i) Let F be an infinite field. Deduce from Sheet 3 Question 4(i) that $J(F[t_1, \ldots, t_k])$ is zero.

   (ii) Show that if $R \subseteq S$ is an integral extension then $J(S) \cap R = J(R)$. Deduce that if, in addition, S is an integral domain, then $J(S) = \{0\}$ if and only if $J(R) = \{0\}$.

   (iii) Now let F be an arbitrary field. Using the Noether Normalisation Lemma, deduce that every finitely generated F-algebra is a Jacobson ring.

3. (i) Prove that $\mathbb{Q}$ is not a finitely generated $\mathbb{Z}$-algebra.

   (ii) Let F be a field which is finitely generated as a $\mathbb{Z}$-algebra. Prove that $\mathrm{char}(F) \neq 0$.

   Hint: Suppose that F has characteristic zero. Consider the three rings $\mathbb{Z} \subseteq \mathbb{Q} \subseteq F$.

   (iii) Let S be a finitely generated $\mathbb{Z}$-algebra and M a maximal ideal of S. Prove that $|S/M| < \infty$.

4. Let R be a subring of a field E and Y a multiplicatively closed subset of R with $1 \in Y$ and $0 \notin Y$. Let S be the integral closure of R in E. Prove that the integral closure of $Y^{-1}R$ in E is $Y^{-1}S$.

5. Let R be an integrally closed domain with field of fractions F, let $E \supseteq F$ be an algebraic field extension and let $a \in E$. Show a is integral over R if and only if the (monic) minimal polynomial of a over F lies in $R[t]$.

   Hint: consider a suitable splitting field.

   Does this necessarily hold if R is not integrally closed?

6. Let R be an integrally closed, Noetherian, local, integral domain of dimension 1, with unique maximal ideal P. Using the steps below, or otherwise, prove that R is a principal ideal domain.

   (i) Let $0 \neq a \in P$. Show that for some $n \geq 1$ we have $P^{n-1} \nsubseteq aR$ and $P^n \subseteq aR$, where $P^0 := R$. Let $b \in P^{n-1} \backslash aR$ and put $y = a^{-1}b$. Show that if $yP \subseteq P$ then $y \in R$. Deduce in fact $yP \nsubseteq P$.

   Hint: consider the action of y on the R-module P.

   (ii) Now deduce that $yP = R$ and hence that P is a principal ideal.

   (iii) Let I be a proper, non-zero ideal of R. Prove that $I = P^n$ for some $n \geq 1$.

   Hint: first show that there is a maximal n for which $I \subseteq P^n$.

7. Let R be a ring, not necessarily Noetherian. Let P be a prime ideal of $S = R[t]$ with $t \in P$. Show that if $h(P/tS)$ is finite then $h(P) > h(P/tS)$.

   Hint: show that if Q is a prime ideal of R, then QS is prime in S.

   Deduce that if $\dim(R)$ is finite then $\dim(S) > \dim(R)$.