

### Problem Set 3

1. Prove that there is no descending sequence  $X_0 \ni X_1 \ni \dots$  of sets, that is, there is no function  $f$  with domain  $\omega$  such that  $f(n^+) \in f(n)$  for all  $n \in \omega$ . Hint: Apply the Axiom of Foundation to a suitably chosen set.

2. Use the Axiom of Foundation to show that, if  $A$  is a non-empty set, then  $A \neq A \times A$ . Hint: Apply the Axiom of Foundation to a suitably chosen set.

3. Prove that a subset of a finite set is finite. Hint: First show, by induction, that, for  $n \in \omega$ , every subset of  $n$  is equinumerous with some natural number. (A set is defined to be finite if it is equinumerous with an element of  $\omega$ .)

4. Prove that the following properties of a set  $X$  are equivalent:

(1)  $\omega \preceq X$  (i.e. there is an injective function  $f : \omega \rightarrow X$ )

(2) there exists a function  $g : X \rightarrow X$  which is injective but not surjective.

Hint: For (2) $\Rightarrow$ (1) use the Recursion Theorem, and induction to verify that the function you define is indeed injective.

5. Suppose  $\kappa, \lambda, \mu$  are cardinals. Prove (no need to check obvious bijections)

(i)  $(\kappa + \lambda) + \mu = \kappa + (\lambda + \mu)$

(ii)  $(\kappa \cdot \lambda) \cdot \mu = \kappa \cdot (\lambda \cdot \mu)$

(iii)  $\kappa \cdot (\lambda + \mu) = \kappa \cdot \lambda + \kappa \cdot \mu$

(iv)  $\kappa^{\lambda + \mu} = \kappa^\lambda \cdot \kappa^\mu$

(v)  $\kappa^{\lambda \cdot \mu} = (\kappa^\lambda)^\mu$

(vi)  $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$

6. (a) Let  $A, X, Y$  be sets such that  $X \preceq A$ . Prove that  $X^Y \preceq A^Y$ . Deduce that, for cardinals  $\kappa, \lambda, \mu$ , if  $\kappa \leq \lambda$  then  $\kappa^\mu \leq \lambda^\mu$ .

(b) Now let  $A, B, X, Y$  be sets with  $X \preceq A$  and  $Y \preceq B$ . Prove that, apart from exceptional case(s),  $X^Y \preceq A^B$ . [You need to show that the map you give from  $X^Y$  to  $A^B$  is really injective.] What are the exceptional cases?

7. Calculate the cardinalities of the following sets, simplifying your answers as far as possible: your answer in each case should be a cardinal from the list  $\aleph_0, 2^{\aleph_0}, 2^{2^{\aleph_0}}, \dots$

(i) the set of all finite sequences of natural numbers [Note that the axioms given so far do not prove that a countable union of countable sets is countable. Use unique factorization of non-zero natural numbers into powers of primes.]

**PTO**

- (ii) the set of functions  $f : \mathbb{R} \rightarrow \mathbb{R}$
- (iii) The set of continuous functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  Hint: a continuous function is determined by its values on  $\mathbb{Q}$ .
- (iv) The set of equivalence relations on  $\omega$ . Hint: To get a lower bound think about partitions of  $\omega$ .

8. Let  $f : X \rightarrow Y$  be surjective. Prove that  $\mathcal{P}(Y) \preceq \mathcal{P}(X)$ . [*You should not assume there exists an injective map  $g : Y \rightarrow X$  as the axioms we have so far do not suffice to prove this.*]

9. (a) Let  $\kappa$  be any cardinal number and  $n \in \omega$ . Prove that (for cardinal addition)

- (i)  $\kappa + 0 = \kappa$  and  $\kappa \cdot 0 = 0$
- (ii)  $\kappa \cdot n^+ = \kappa \cdot n + \kappa$

(b) We now have two definitions of addition and multiplication for elements of  $\omega$ . Prove that they agree.