Combinatorics C8.3 **Problems 1** **JL MT2019**

MFoCS questions should be done by MFoCS students, although everyone is encouraged to try them; we may not have time to go through them in classes. Questions (or parts of questions) marked with a + sign are intended as a challenge for enthusiasts: we will not go through them in classes!

1. Write down all antichains contained in $\mathcal{P}(1)$ and $\mathcal{P}(2)$. How many different antichains are there in $\mathcal{P}(3)$?

2. Let $k \leq n/2$, and suppose that $\mathcal{F}$ is an antichain in $\mathcal{P}(n)$ such that every $A \in \mathcal{F}$ has $|A| \leq k$. Prove that $|\mathcal{F}| \leq \binom{n}{k}$.

3. Let $(P, \leq)$ be a poset. Suppose that every chain in P has at most k elements. Prove that P can be written as the union of k antichains.

4. Suppose $\mathcal{F} \subset \mathcal{P}(n)$ is a set system containing no chain with $k+1$ sets.

   (a) Prove that
   $$\sum_{i=0}^{n} \frac{|\mathcal{F}_i|}{\binom{n}{i}} \leq k,$$
   where $\mathcal{F}_i = \mathcal{F} \cap [n]^{(i)}$ for each i.

   (b) What is the maximum possible size of such a system?

5. (a) Look up Stirling's Formula. Use it to find an asymptotic estimate of form $(1+o(1))f(n)$ for $\binom{n}{n/2}$.

   (b) Now do the same for $\binom{n}{pn}$ where $p \in (0,1)$ is a constant. Write your answer in terms of the binary entropy function $H(p) = -p\log p - (1-p)\log(1-p)$.

6. (MFoCS) Let $f(n)$ be the number of subsets $\mathcal{A} \subset \mathcal{P}(n)$ such that $\mathcal{A}$ is an antichain.

   (a) Prove that $f(n) \geq 2^{\binom{n}{\lfloor n/2 \rfloor}}$ for every n.

   (b) Prove that, for every $\epsilon > 0$, $f(n) \leq 2^{\epsilon 2^n}$ for all sufficiently large n.

   (c)$^+$ Prove that there is a constant $C > 0$ such that
   $$f(n) \leq C^{\binom{n}{\lfloor n/2 \rfloor}}$$
   for every n.

7. Let $\mathcal{A}$ be an antichain not of the form $[n]^{(r)}$. Must there exist a maximal chain disjoint from $\mathcal{A}$?

8. (Another proof of Sperner's Lemma) Given antichains $\mathcal{A}$ and $\mathcal{B}$ we write

$$\mathcal{A} \vee \mathcal{B} = \{X \in \mathcal{A} \cup \mathcal{B} \text{ such that } \forall\, Y \in \mathcal{A} \cup \mathcal{B} \text{ we have } X \not\subsetneq Y\}$$

for the set of inclusion-maximal elements of $\mathcal{A} \cup \mathcal{B}$, and

$$\mathcal{A} \wedge \mathcal{B} = \{X \in \mathcal{A} \cup \mathcal{B} \text{ such that } \forall\, Y \in \mathcal{A} \cup \mathcal{B} \text{ we have } Y \not\subsetneq X\}$$

for the set of inclusion-minimal elements of $\mathcal{A} \cup \mathcal{B}$.

(a) Show that
$$\mathcal{A} \cap \mathcal{B} \subseteq (\mathcal{A} \vee \mathcal{B}) \cap (\mathcal{A} \wedge \mathcal{B})$$
and
$$\mathcal{A} \cup \mathcal{B} = (\mathcal{A} \vee \mathcal{B}) \cup (\mathcal{A} \wedge \mathcal{B}).$$

(b) Let m be the maximal size of an antichain. By using part (a), show that if $\mathcal{A}$ and $\mathcal{B}$ are antichains of size m then so are $\mathcal{A} \vee \mathcal{B}$ and $\mathcal{A} \wedge \mathcal{B}$.

(c) Let $\Gamma = \{\mathcal{A}_1, \dots, \mathcal{A}_K\}$ be the set of all antichains of size m, and let $\mathcal{A} = \mathcal{A}_1 \vee (\mathcal{A}_2 \vee (\cdots \vee (\mathcal{A}_{K-1} \vee \mathcal{A}_K) \dots))$. Show that $\mathcal{A}$ is fixed by permutations of the ground set $[n]$; that is, if σ is a permutation of $[n]$ and we write $\sigma(A) = \{\sigma(a) : a \in A\}$ and $\sigma(\mathcal{A}) = \{\sigma(A) : A \in \mathcal{A}\}$, we have that $\sigma(\mathcal{A}) = \mathcal{A}$.

(d) Deduce that the maximum size m of an antichain is $\binom{n}{\lfloor n/2 \rfloor}$.

9. We say that a family $\mathcal{A}$ *splits* $[n]$ if for every distinct $i, j \in [n]$ there exists $A \in \mathcal{A}$ such that $i \in A$ and $j \notin A$. Find a tight upper bound for n in terms of $|\mathcal{A}|$.

10. We say that a chain is symmetric if it is of the form $A_k \subset A_{k+1} \subset \cdots \subset A_{n-k}$ where $A_r \in [n]^{(r)}$. In this question we will partition $\mathcal{P}(n)$ into symmetric chains (which, of course, implies Sperner's lemma).

(a) Consider a bipartite graph G with vertex partitions given by $X = [n]^{(k)} \cup [n]^{(n-k-1)}$ and $Y = [n]^{(n-k)} \cup [n]^{(k+1)}$ for some $0 \leq$

$k \leq \lfloor n/2 \rfloor - 1$, with edges as follows. Between $[n]^{(k)}$ and $[n]^{(k+1)}$ and between $[n]^{(n-k)}$ and $[n]^{(n-k-1)}$ we have edges as found in the discrete cube Q_n, and between $[n]^{(k+1)}$ and $[n]^{(n-k-1)}$ we place an arbitrary perfect matching. Show that G satisfies the conditions of Hall's marriage theorem and thus contains a perfect matching.

(b) Deduce that $\mathcal{P}(n)$ has a decomposition into symmetric chains.