Combinatorics C8.3 **Problems 0** **EL MT2018**

The aim of this sheet is to highlight some useful results and arguments in combinatorics. The problems make use of a small amount of graph theory, but everything required can be found in the subsection on Hall's theorem in Section 7 of the Part B Graph Theory notes. There will not be tutorials on this problem sheet, but complete solutions are available on the course webpage. All graphs below are assumed to be finite.

Remark: The combinatorics course is essentially self-contained, so do not feel put off if there is notation below which you haven't encountered before.

1. Let G be a bipartite graph with bipartition (A, B). Suppose also that every vertex in G has the same degree $d > 0$.
   (a) Show that $|A| = |B|$.
   (b) Look up Hall's theorem. Use this result to prove that G contains a complete matching.
   (c) Show that the edge set of G can be partitioned into d edge disjoint complete matchings.

2. Let $[n]^{(i)} := \{A \subset \{1, \ldots, n\} : |A| = i\}$ and suppose that $i < n/2$. Prove that for each $A \in [n]^{(i)}$ we can choose a set $B_A \in [n]^{(i+1)}$ so that $A \subset B_A$ for all A and such that the sets $\{B_A\}_A$ are all distinct.

3. Let G be a bipartite graph with bipartition (A, B) which contains a complete matching from A to B. Prove that there is $a \in A$ such that every edge $ab \in E(G)$ lies in a complete matching from A to B.

   Hint: Read the 'direct proof' of Hall's theorem in the Part B notes.

4. Let $\mathcal{P}[n]$ denote the power set of $[n] := \{1, \ldots n\}$.
   (a) Prove that $|\mathcal{P}[n]| = 2^n$.
   (b) Suppose a set $A \in \mathcal{P}[n]$ is selected uniformly at random. Let X denote the random variable given by $X(A) := |A|$. Prove that $\mathbb{E}(X) = n/2$ and $\mathbb{V}ar(X) = n/4$.
   (c) Use Chebyshev's inequality and (b) to show that given $\epsilon > 0$ there is $C > 0$ such that $(1 - \epsilon)2^n$ sets $A \subset [n]$ satisfy $\left||A| - \frac{n}{2}\right| \leq C\sqrt{n}$.