

**STOCHASTIC DIFFERENTIAL EQUATIONS**  
**MATH C8.1 - 2019 - SHEET 1**

- (i) Let  $(B_t)_{t \in [0, \infty)}$  be a standard Brownian motion on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Prove using the properties of Brownian motion that the following three processes are martingales.

(a) The process

$$(B_t)_{t \in [0, \infty)}.$$

(b) The process

$$(B_t^2 - t)_{t \in [0, \infty)}.$$

(c) And for every  $\alpha \in \mathbb{R}$ , the process

$$\left( \exp \left( \alpha B_t - \frac{\alpha^2 t}{2} \right) \right)_{t \in [0, \infty)}.$$

- (ii) Let  $(B_t)_{t \in [0, \infty)}$  be a standard Brownian motion on  $\mathbb{R}^2$  and for  $R \in (0, \infty)$  let  $B_R$  denote the ball of radius  $R$  centered at the origin. For every  $t \in (0, \infty)$ , compute

$$\mathbb{P}[B_t \in B_R],$$

and thereby prove that

(i)  $\mathbb{P}(B_t \notin B_{\sqrt{2\lambda t}}) = e^{-\lambda}$ ,

(ii) and that, for the Lebesgue measure  $|B_R|$  of  $B_R$ ,

$$\lim_{R \rightarrow 0} \frac{\mathbb{P}[B_t \in B_R]}{|B_R|} = \frac{1}{2\pi t}.$$

What happens in dimension three?

- (iii) Let  $(B_t)_{t \in [0, \infty)}$  be a standard Brownian motion on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .

(i) Prove that  $W_t = tB_{\frac{1}{t}}$  is a standard Brownian motion on  $(\Omega, \mathcal{F}, \mathbb{P})$ .

(ii) For every  $a \in [0, \infty)$ , let  $T_a$  denote the stopping time

$$T_a = \inf\{t \in [0, \infty) : B_t = a\}.$$

Show using the reflection principle that

$$\mathbb{P}(T_a < t) = 2\mathbb{P}(B_t > a),$$

and thereby deduce that the probability density function  $f_a$  of  $T_a$  on  $[0, \infty)$  is

$$f_a(t) = \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{a^2}{2t}\right).$$

(iii) For every  $a \in [0, \infty)$ , define the random time

$$S_a = \sup\{t \in [0, \infty) : B_t = at\}.$$

Is  $S_a$  a stopping time? Show that  $S_a = 1/T_a$  in distribution and find  $\mathbb{E}[S_a]$  and  $\mathbb{E}[B_{B_{S_a}}]$ .

- (iv) A real-valued centered Gaussian process is a real valued process  $(X_t)_{t \in [0, \infty)}$  on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with finite dimensional distributions that are normally distributed and mean zero. A centered Gaussian process  $(X_t)_{t \in [0, \infty)}$  is a fractional Brownian motion with Hurst parameter  $h \in (0, 1)$  if  $\mathbb{P}[X_0 = 0] = 1$  and if, for every  $s, t \in [0, \infty)$ ,

$$\mathbb{E}[X_t X_s] = \frac{1}{2} (t^{2h} + s^{2h} - |t - s|^{2h}).$$

- (a) Show that, for every  $s \leq t \in [0, \infty)$ , the increment  $X_t - X_s$  has mean zero and variance  $|t - s|^{2h}$ .  
 (b) Show that for every  $p \in (0, \infty)$  there exists  $c_p \in (0, \infty)$  such that, for every  $s \leq t \in [0, \infty)$ ,

$$\mathbb{E}[|X_t - X_s|^p] = c_p |t - s|^{hp}.$$

- (c) Show that fractional Brownian motion has a continuous modification, and determine its Hölder exponent. (Hint: Use Kolmogorov's continuity criterion.)  
 (d) For each  $h \in (0, 1)$ , determine the value  $p \in (0, \infty)$  so that fractional Brownian motion with Hurst parameter  $h$  has finite and nonzero  $p$ -variation. Is fractional Brownian motion a semimartingale for any  $h \in (0, 1)$ ?

- (v) Let  $(M_t)_{t \in [0, \infty)}$  be a continuous local martingale with  $M_0 = 0$ .

- (a) Prove that if  $\mathbb{E}[\langle M \rangle_\infty] < \infty$  then  $M$  is  $L^2$ -bounded in the sense that

$$\sup_{t \in [0, \infty)} \mathbb{E}[M_t^2] < \infty.$$

In this case, conclude that the process  $(M_t^2 - \langle M \rangle_t)_{t \in [0, \infty)}$  is uniformly integrable.

- (b) Prove that  $M$  converges almost surely as  $t \rightarrow \infty$  on the set  $\{\langle M \rangle_\infty < \infty\}$ .

- (vi) Suppose that  $(M_t)_{t \in [0, \infty)}$  is a bounded continuous martingale with finite variation. Prove that

$$M_t^2 = M_0^2 + 2 \int_0^t M_s dM_s,$$

where the final integral is almost surely well-defined. (Hint: Cite here known results about one-dimensional functions of bounded variation.) Deduce that  $M$  is almost surely constant.