STOCHASTIC DIFFERENTIAL EQUATIONS
MATH C8.1 - 2019 - SHEET 4

1. Sheet 4

(i) Let $(M_t)_{t\in[0,\infty)}$ be a continuous local martingale that vanishes at zero. Let $(\mathcal{E}(M)_t)_{t\in[0,\infty)}$ denote the stochastic exponential.
  (a) Show that $(\mathcal{E}(M)_t)_{t\in[0,\infty)}$ is a nonnegative continuous local martingale.
  (b) Show that $(\mathcal{E}(M)_t)_{t\in[0,\infty)}$ is a supermartingale with $\mathbb{E}[\mathcal{E}(M)_t)] \leq 1$ for every $t \in [0,\infty)$.
  (c) Show that $(\mathcal{E}(M)_t)_{t\in[0,\infty)}$ is a continuous martingale if and only if $\mathbb{E}[\mathcal{E}(M)_t)] = 1$ for every $t \in [0,\infty)$.

(ii) The following is Kazamaki's criterion. Let $(L_t)_{t\in[0,\infty)}$ be a continuous local martingale. Prove that if $(\exp(\frac{1}{2}L_t))_{t\in[0,\infty)}$ is a uniformly integrable submartingale, then the stochastic exponential $(\mathcal{E}(L)_t)_{t\in[0,\infty)}$ is a uniformly integrable martingale.
Hint: Show for every $\alpha \in (0,1)$ that

$$\mathcal{E}(\alpha L_t) = (\mathcal{E}(L)_t)^{\alpha^2}\left(Z_t^\alpha\right)^{1-\alpha^2},$$

for $Z_t^\alpha = \exp(\frac{\alpha}{1+\alpha}L_t)$. Use Hölder's inequality, Question (i), and the optional stopping theorem to prove that, for every stopping time T, for every $A \in \mathcal{F}$,

$$\mathbb{E}\left[\mathcal{E}(\alpha L_T)\mathbf{1}_A\right] \leq \mathbb{E}\left[Z_T^\alpha \mathbf{1}_A\right]^{1-\alpha^2}.$$

Conclude using the assumption and $\alpha \in (0,1)$ that $(Z_t^\alpha)_{t\in[0,\infty)}$ is a uniformly integrable submartingale, and therefore that the familly

$$\{\mathcal{E}(\alpha L_T))\colon T \text{ is a stopping time }\} \text{ is a uniformly integrable,}$$

and therefore that $(\mathcal{E}(\alpha L_t))_{t\in[0,\infty)}$ is a uniformly integrable martingale. Hence, for every $\alpha \in (0,1)$,

$$1 = \mathbb{E}[\mathcal{E}(\alpha L_\infty)] \leq \mathbb{E}\left[Z_\infty^\alpha\right]^{1-\alpha^2}.$$

Use the assumption and the dominated convergence theorem to pass to the limit $\alpha \to 1$ to conclude that

$$\mathbb{E}[\mathcal{E}(L)_\infty] = 1,$$

and therefore using Question (i) conclude that $(\mathcal{E}(L)_t)_{t\in[0,\infty)}$ is a martingale.

(iii) The following is Novikov's criterion. Prove that if $(L_t)_{t\in[0,\infty)}$ is a continuous local martingale which satisfies

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\langle L\rangle_\infty\right)\right] < \infty,$$

then $(\mathcal{E}(L))_{t\in[0,\infty)}$ is a uniformly integrable martingale.
(Hint: Show that Novikov's criterion implies Kazamaki's criterion. First use the fact that the exponential integrability implies for every $p \in (0,\infty)$ that

$$\mathbb{E}\left[\langle L\rangle_\infty^{\frac{p}{2}}\right] < \infty.$$

Use this fact and the Burkholder-Davis-Gundy inequalities to prove that $(L_t)_{t\in[0,\infty)}$ is a uniformly integrable martingale. Then use the equality

$$\exp\left(\frac{1}{2}L_\infty\right) = \mathcal{E}(L)_\infty^{\frac{1}{2}} \exp\left(\frac{1}{4}\langle L\rangle_t\right),$$

Hölder's inequality, Question (i), and the assumptions to conclude that

$$\mathbb{E}\left[\exp\left(\frac{1}{2}L_\infty\right)\right] < \infty.$$

Use this fact to conclude that $(\exp(\frac{1}{2}L_t))_{t\in[0,\infty)}$ is a uniformly integrable submartingale, and then apply Kazamaki's criterion.

(iv) Let $(B_t)_{t\in[0,\infty)}$ be a standard one-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to a filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$. Suppose that $b\colon \mathbb{R}\to\mathbb{R}$ is a bounded, measurable function. Define measures $\{\mathbb{Q}_T\}_{T\in(0,\infty)}$ on $\{(\Omega, \mathcal{F}_T, \mathbb{P})\}_{T\in(0,\infty)}$ which satisfy the following three properties.

(a) The measures are compatible in the sense that $\mathbb{Q}_{T_1} = \mathbb{Q}_{T_2}$ on $\mathcal{F}_{T_1}$ for every $T_1 \leq T_2 \in [0,\infty)$.

(b) For every $T\in[0,\infty)$ the measure $\mathbb{Q}_T$ is mutually absolutely continuous with respect to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T, \mathbb{P})$.

(c) The process $(\tilde{B}_t)_{t\in[0,\infty)}$ defined by

$$\tilde{B}_t = B_t - \int_0^t b(B_s)\,\mathrm{d}B_s,$$

is for every $T\in(0,\infty)$ a standard Brownian motion with respect to $(\mathbb{Q}_T, \mathcal{F}_T)$ on $[0,T]$.

Prove that for every $T\in(0,\infty)$ the pair $(B_t, \tilde{B}_t)_{t\in[0,T]}$ is a weak solution to the equation

$$\begin{cases} \mathrm{d}B_t = b(B_t)\,\mathrm{d}t + \mathrm{d}\tilde{B}_t & \text{in } (0,T), \\ B_0 = 0, \end{cases} \tag{1.1}$$

with respect to $(\mathbb{Q}_T, \mathcal{F}_T)$. Deduce that uniqueness in law holds for (1.1).

(v) (Skorokhod's Lemma) Let $y\colon [0,\infty)\to\mathbb{R}$ be a real-valued function that satisfies $y(0)\geq 0$. Prove that there exist unique functions $a, z\colon [0,\infty)\to\mathbb{R}$ which satisfy the following three properties.

(a) We have the decomposition

$$z = y + a.$$

(b) The function z is nonnegative

$$z \geq 0.$$

(c) The function a is increasing, continuous, and vanishes at zero and the corresponding Riemann-Stieltjes measure $\mathrm{d}a$ is supported on the set $\{s\colon z(s) = 0\}$.

Furthermore, the function a is given by

$$a(t) = \left[\sup_{s\in[0,t]} (-y(s))\right] \vee 0.$$

(vi) Let $(B_t)_{t\in[0,\infty)}$ be a standard one-dimensional Brownian motion. Define the process $(X_t)_{t\in[0,\infty)}$ by

$$X_t = \int_0^t \operatorname{sgn}(B_s)\,\mathrm{d}B_s.$$

(a) Show that $(X_t)_{t\in[0,\infty)}$ is a standard Brownian motion with respect to the filtration

$$\mathcal{F}_t^{|B|} = \sigma(|B_s| : s \in [0,t]).$$

Furthermore, observe that $\mathcal{F}_t^{|B|} \subsetneq \mathcal{F}_t^B$.

(b) Let $(L_t^0)_{t\in[0,\infty)}$ denote the local time of $(B_t)_{t\in[0,\infty)}$ at zero. Prove that

$$L_t^0 = \sup_{s\in[0,t]} (-X_s)\,.$$

(c) Let $(S_t)_{t\in[0,\infty)}$ be defined by $S_t = \sup_{s\in[0,t]} |B_s|$. Show that the two-dimensional processes $(S_t - B_t, S_t)$ and $(|B_t|\,, L_t^0)$ have the same law.

(d) For every $a \in \mathbb{R}$ let $(L_t^a)_{t\in[0,\infty)}$ denote the local time of $(B_t)_{t\in[0,\infty)}$ at $a \in \mathbb{R}$. Deduce for every $a \in \mathbb{R}$ that $\mathbb{P}[L_\infty^a = \infty] = 1$.

(vii) Let $(B_t)_{t\in[0,\infty)}$ be a standard one-dimensional Brownian motion. Prove that the stochastic differential equation

$$\begin{cases} \mathrm{d}X_t = \operatorname{sgn}(X_t)\,\mathrm{d}B_t \quad \text{in } (0,\infty), \\ X_0 = 0, \end{cases} \tag{1.2}$$

had a weak solution but no strong solution. Deduce that uniqueness in law holds for equation (1.2) but that pathwise uniqueness does not hold.

(viii) Let $\alpha \in (0, 1/2)$, let $\sigma\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$\sigma(x) = |x|^\alpha \wedge 1,$$

and let $(B_t)_{t\in[0,\infty)}$ be a standard one-dimensional Brownian motion. Show that the map

$$t \in [0,\infty) \to \int_0^t \sigma^{-2}(B_s)\,\mathrm{d}s,$$

is almost surely finite. Let $(\tau_t)_{t\in[0,\infty)}$ denote the associated time-changes

$$\tau_t = \inf\left\{ s \in [0,\infty)\colon \int_0^s \sigma^{-2}(B_r)\,\mathrm{d}r = t \right\}.$$

Show that $X_t = B_{\tau_t}$ and $X_t = 0$ are two solutions of the equation

$$\begin{cases} \mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}B_t \quad \text{in } (0,\infty), \\ X_0 = 0. \end{cases}$$

Conclude that uniqueness in law does not hold for this equation, despite the fact that the second of these solutions is a strong solution.