Problem sheet 0

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All questions use natural units, where Newton's constant G and the speed of light c are both equal to 1.

The problems in this set do not require any specialist knowledge but should get you thinking about the kinds of issues which will be important in studying general relativity. The solutions are provided below the questions – but please attempt the questions before looking at the solutions!

Questions/comments/correction to Joseph.Keir@maths.ox.ac.uk

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1. *Newtonian black holes*

Suppose that there is a spherically symmetric star of radius R and mass M. Let a small particle of mass m be ejected from the surface of the star in a radial direction at a velocity v.

a) Show that, if $v^2 < \frac{2M}{R}$ then the maximum radius that the particle reaches is

$$r_{\max} = \frac{2M}{\frac{2M}{R} - v^2}$$

Note that this is *independent* of the mass of the escaping particle m. What happens if $v^2 \geq \frac{2M}{R}$?

b) The velocity $v^2 = \frac{2M}{R}$ is called the *escape velocity*. Now suppose that the particle is actually a photon, which leaves the star travelling at the speed of light $v = 1$. Treating the photon as a particle of mass m, and then taking $m \to 0$, show that, if $R < 2M$ then light cannot escape the star.

This idea for a "dark star" was originally proposed in 1783. Amazingly, despite using Newtonian gravity and assuming that light can slow down, the radius $R = 2M$ turns out to be the correct radius for a black hole of mass M in Einstein's general relativity!

2. *Length contraction and time dilation*

In this question we work in the framework of special relativity.

a) In the distant future, the railway company HS1000 Ltd. wants to build a railway connected Oxford to Cambridge (in a straight line). The company executives are keen to save money by laying down as little track as possible, and for this reason they decide not to lay two parallel tracks for the entire length of the route to allow trains to run freely in both directions. Instead, they will lay down a single track, and then, in the middle of the route, they will build a small section of parallel track, joined at both ends to the main line: this is called a *passing siding* or *passing loop* (see figure 1).

The trains running on this track are of length L. In a bid to save even more money, it is decided that the siding will be of length $\frac{1}{2}L$, i.e. half the length of the trains. The siding should be long enough for trains to pass each other when travelling in opposite directions. What is the minimum speed, v, at which the trains must run in order to be able to pass each other on the sidings?

b) One annoying aspect of travelling on this train is that, at the end of your journey, you have to readjust your watch. Suppose that Alice takes a journey on the train, while Bob remains in Oxford,

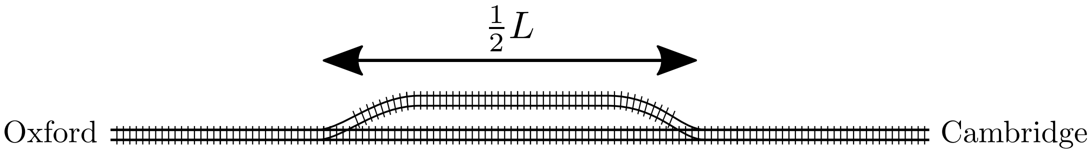

Figure 1

and suppose that, according to Alice's watch, the journey takes a time T. Before the journey, Alice and Bob's watches were synchronised, and at the end of the journey Alice comes to a complete stop. If Alice then returns on the next train and then compares her watch with Bob's, by how much will Alice need to adjust her watch - and in which direction - so that it again matches the time on Bob's watch?

For both parts **a)** and **b)**, it might be helpful to recall the formula for a Lorentz boost: if observer A measures time and space in coordinates (t, x, y, z), and observer B moves past observer A at a speed v along the x axis (with the same spatial orientation) then observer B will measure using coordinates (t', x', y', z')

$$\begin{aligned} t' &= \frac{1}{\sqrt{1-v^2}}(t - vx) \\ x' &= \frac{1}{\sqrt{1-v^2}}(x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

3. *Relativity of simultaneity*

Now suppose that Bob stands on a platform somewhere in the middle of the route (say, at Milton Keynes) where the train doesn't stop, while Alice takes a seat in the dead centre of the train. At the exact moment when Alice passes Bob, he takes a photo of her from the platform, using a camera with a powerful flash.

According to Alice, how long will it take the light from the camera flash to reach the front of the train? How long will it take to reach the back of the train? Does the light reach the front of the train first, or the back?

Answer the same questions from Bob's point of view.

4. *Maxwell's equations and the electromagnetic field tensor*

Maxwell's equations (in the vacuum and in natural units) are

$$\begin{aligned} \operatorname{div}\mathbf{E} &= 0 \\ \operatorname{curl}\mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 \\ \operatorname{div}\mathbf{B} &= 0 \\ \operatorname{curl}\mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} &= 0 \end{aligned}$$

Let use write

$$\partial_t := \frac{\partial}{\partial t}\Big|_{x,y,z} \qquad \partial_{t'} := \frac{\partial}{\partial t'}\Big|_{x',y',z'}$$

$$
\begin{aligned}
\partial_x &:= \left.\frac{\partial}{\partial x}\right|_{t,y,z} & \partial_{x'} &:= \left.\frac{\partial}{\partial x'}\right|_{t',y',z'} \\
\partial_y &:= \left.\frac{\partial}{\partial y}\right|_{t,x,z} & \partial_{y'} &:= \left.\frac{\partial}{\partial y'}\right|_{t',x',z'} \\
\partial_z &:= \left.\frac{\partial}{\partial z}\right|_{t,x,y} & \partial_{z'} &:= \left.\frac{\partial}{\partial z'}\right|_{t',x',y'}
\end{aligned}
$$

where the coordinates (t, x, y, z) and (t', x', y', z') are related as in question 2.

a) Show that these operators are related by

$$
\begin{aligned}
\partial_{t'} &= \gamma\partial_t + \gamma v\partial_x \\
\partial_{x'} &= \gamma v\partial_t + \gamma\partial_x \\
\partial_{y'} &= \partial_y \\
\partial_{z'} &= \partial_z
\end{aligned}
$$

b) Now show that, if we write

$$
\begin{aligned}
(E')^{x'} &= E^x \\
(E')^{y'} &= \gamma E^y - \gamma v B^z \\
(E')^{z'} &= \gamma E^z + \gamma v B^y \\
(B')^{x'} &= B^x \\
(B')^{y'} &= \gamma B^y + \gamma v E^z \\
(B')^{z'} &= \gamma B^z - \gamma v E^y
\end{aligned}
$$

Then the Maxwell equations, written in the new coordinates and with respect to these fields take the form

$$
\begin{aligned}
\operatorname{div}' \mathbf{E}' &= 0 \\
\operatorname{curl}' \mathbf{E}' + \frac{\partial \mathbf{B}'}{\partial t'} &= 0 \\
\operatorname{div}' \mathbf{B}' &= 0 \\
\operatorname{curl}' \mathbf{B}' - \frac{\partial \mathbf{E}'}{\partial t'} &= 0
\end{aligned}
$$

where, for example,

$$
\begin{aligned}
\operatorname{div}' \mathbf{E}' &= \partial_{x'}(E')^{x'} + \partial_{y'}(E')^{y'} + \partial_{z'}(E')^{z'} \\
(\operatorname{curl}' \mathbf{E}')^{x'} &= \partial_{y'}(E')^{z'} - \partial_{z'}(E')^{y'}
\end{aligned}
$$

c) Now define the *electromagnetic field tensor*

$$
F_{ab} = \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & -B^z & B^y \\ -E^y & B^z & 0 & -B^x \\ -E^z & -B^y & B^x & 0 \end{pmatrix}
$$

and its *Hodge dual* is

$$
*F_{ab} = \begin{pmatrix} 0 & B^x & B^y & B^z \\ -B^x & 0 & E^z & -E^y \\ -B^y & -E^z & 0 & E^x \\ -B^z & E^y & -E^x & 0 \end{pmatrix}
$$

Write the Lorentz transformation as

$$(x')^{a'} = \Lambda_b{}^{a'} x^b$$

$$\Lambda_b{}^{a'} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

note that the inverse of this is given by

$$(\Lambda^{-1})_{a'}{}^{b} = \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

i) Show that Maxwell's equations can be written as

$$\partial^a F_{ab} = 0$$
$$\partial^a * F_{ab} = 0$$

(remember that $\partial^a = m^{ab}\partial_b$, where $m^{ab} = \mathrm{diag}(-1, 1, 1, 1)$)

ii) Define the tensor

$$F'_{a'b'} := (\Lambda^{-1})_{a'}{}^{c} (\Lambda^{-1})_{b'}{}^{d} F_{cd}$$

and show that

$$F'_{a'b'} = \begin{pmatrix} 0 & (E')^{x'} & (E')^{y'} & (E')^{z'} \\ -(E')^{z'} & 0 & -(B')^{z'} & (B')^{y'} \\ -(E')^{y'} & (B')^{z'} & 0 & -(B')^{x'} \\ -(E')^{z'} & -(B')^{y'} & (B')^{x'} & 0 \end{pmatrix}$$

iii) The components of "Spacetime vectors" transform under a Lorentz transform as

$$(V')^{a'} = \Lambda_b{}^{a'} V^b$$

(consider, for example, the displacement vector x^a). On the other hand, "spacetime covectors" transform as

$$(V')_{a'} = (\Lambda^{-1})_{a'}{}^{b} V_b$$

Can the electric and magnetic fields be considered vectors or covectors in this sense?

5. *Fictitious forces*

Consider a free particle obeying Newton's laws, so that, in inertial coordinates,

$$\ddot{x}^i = 0$$

where the dots represent time derivatives.

Change coordinates to $y^i = y^i(t, x)$. Show that the equations of motion become

$$\ddot{y}^i = \frac{\partial^2 y^i}{\partial x^j \partial x^k} \dot{x}^j \dot{x}^k + 2 \frac{\partial^2 y^i}{\partial x^j \partial t} \dot{x}^j + \frac{\partial^2 y^i}{\partial t^2}$$

Now consider Newton's law $F = ma$ in the coordinates y^i, interpreting the acceleration as $a^i = \ddot{y}^i$. Show that the particle experiences a 'force' given by

$$F^i = m \left(\frac{\partial^2 y^i}{\partial x^j \partial x^k} \dot{x}^j \dot{x}^k + 2 \frac{\partial^2 y^i}{\partial x^j \partial t} \dot{x}^j + \frac{\partial^2 y^i}{\partial t^2} \right)$$

This ‘force’ is called a *fictitious force* - it is not a real force, but simply an artefact due to the fact that the coordinates y^i are not inertial coordinates. An example of such a force is the ‘centrifugal force’.

Note that all fictitious forces must be proportional to the mass of the particle on which they act. Can you think of any other force which is proportional to the mass of the object on which it acts?

The solutions are provided on the next page. Do not look unless you've attempted the questions!

Solutions

1. a) The total energy of the particle is

$$\frac{1}{2}m\dot{r}^2 - \frac{mM}{r}$$

Equating the initial energy and the energy at $r = r_{\text{max}}$:

$$\begin{aligned} \frac{1}{2}mv^2 - \frac{mM}{R} &= -\frac{mM}{r_{\text{max}}} \\ \Rightarrow r_{\text{max}} &= \frac{2M}{\frac{2M}{R} - v^2} \end{aligned}$$

If $v^2 \geq \frac{2M}{R}$ then the particle escapes to infinity.

b) The escape velocity is $\sqrt{\frac{2M}{R}}$. If this is greater than 1, then photons which leave at a velocity $v = 1$ travel below the escape velocity, and so will fall back to the 'star'.

2. a) Suppose that the train travels in the x direction. From the point of view of someone on the train, say someone sitting at the back of the train, the back of the train follows the worldline $(t, x, y, z) = (t, 0, 0, 0)$ and the front of the train is at $(t, L, 0, 0)$.

Using a Lorentz boost to change frame to an observer in the rest frame of the train tracks (i.e. a 'stationary' observer) means performing a Lorentz boost with velocity $-v$ (from the point of view of the train, this new observer moves backwards) and so, in this frame, the back and front of the train follow the worldlines

$$\begin{aligned} (t', x', y', z') &= (\gamma t, \gamma v t, 0, 0) \qquad \text{(back)} \\ (t', x', y', z') &= (\gamma(t + vL), \gamma(L + vt), 0, 0) \qquad \text{(front)} \end{aligned}$$

where $\gamma = \frac{1}{\sqrt{1-v^2}}$.

Suppose that this stationary observer sees the back of the train passing the point $x' = x'_0$ at a time $t' = t'_0$, which corresponds to the start of the passing loop. Then $x'_0 = vt'_0$, so the back of the train is at

$$(t', x', y', z') = (t'_0, x'_0, 0, 0)$$

By solving for t, we also find that, when $t' = t'_0$, the front of the train is at

$$(t', x', y', z') = (t'_0, x'_0 + \gamma^{-1}L, 0, 0)$$

For the train to 'fit', we therefore need

$$\begin{aligned} \gamma^{-1}L &\leq \frac{1}{2}L \\ \Rightarrow \gamma &\geq 2 \\ \Rightarrow v &\geq \frac{\sqrt{3}}{2} \end{aligned}$$

so the minimum speed that the train can travel is $\frac{\sqrt{3}}{2}$ (or $\frac{\sqrt{3}}{2}c$, restoring the speed of light).

b) Suppose Alice is at the back of the train, so she uses coordinates (t, x, y, z). She reaches her Cambridge at $t = T$. In Bob's coordinates (t', x', y', z') Alice has coordinates $(\gamma T, \gamma v T, 0, 0, 0)$ when she arrives in Cambridge. If she immediately returns to Bob then, by symmetry, she will reach Bob at the event $(t', x', y', z') = (2\gamma T, 0, 0, 0)$, and therefore, the amount of time that has passed according to Bob's clock is $2\gamma T$. On the other hand, for Alice the amount of time that has passed is $2T$. Since $\gamma > 1$, she will have to adjust her watch *forward*, by an amount $2T(\gamma - 1)$.

3. From Alice's point of view the flash occurs at the centre of the train, which is at rest. Since the speed of light is constant, light reaches the front of the train and the back at the same time. Since the train is of length L, it will take a time $\frac{2}{L}$ for light to reach both the front and rear of the train.

From Bob's point of view the train is moving with a velocity v. Since the speed of light is constant (and 1, in our units) Bob will see the forward travelling light flash at $x' = t'$ and the backward travelling light at $x' = -t'$. Meanwhile, Bob sees the front and rear of the train at

$$
\begin{aligned}
(t', x', y', z') &= \left(\gamma\left(t + \frac{1}{2}vL\right), \gamma\left(\frac{1}{2}L + vt\right), 0, 0\right) && \text{(front)} \\
(t', x', y', z') &= \left(\gamma\left(t - \frac{1}{2}vL\right), \gamma\left(-\frac{1}{2}L + vt\right), 0, 0\right) && \text{(rear)}
\end{aligned}
$$

(see the previous question). Put another way, from Bob's point of view the front and rear of the train are at

$$
\begin{aligned}
x' &= \frac{1}{2}L\gamma^{-1} + vt' && \text{(front)} \\
x' &= -\frac{1}{2}L\gamma^{-1} + vt' && \text{(rear)}
\end{aligned}
$$

so the light will reach the front and rear of the train at times

$$
\begin{aligned}
t' &= \frac{L}{2}\sqrt{\frac{1+v}{1-v}} && \text{(front)} \\
t' &= \frac{L}{2}\sqrt{\frac{1-v}{1+v}} && \text{(rear)}
\end{aligned}
$$

In particular, since $v > 0$, the light will reach the rear of the train *before* it reaches the front of the train, from Bob's point of view.

4. a) We have

$$
\begin{aligned}
\partial_{t'} &= \left(\left.\frac{\partial t}{\partial t'}\right|_{x',y',z'} \partial_t\right) + \left(\left.\frac{\partial x}{\partial t'}\right|_{x',y',z'} \partial_x\right) + \left(\left.\frac{\partial y}{\partial t'}\right|_{x',y',z'} \partial_y\right) + \left(\left.\frac{\partial z}{\partial t'}\right|_{x',y',z'} \partial_z\right) \\
&= \gamma\partial_t + v\gamma\partial_x
\end{aligned}
$$

The other formulae follow similarly.

b) Calculating

$$
\begin{aligned}
\operatorname{div}' \boldsymbol{E}' &= \partial'_{x'} E^{x'} + \partial'_{y'} E^{y'} + \partial'_{z'} E^{z'} \\
&= \gamma v \partial_t E^x + \gamma \partial_x E^x + \partial_y(\gamma E^y - \gamma v B^z) + \partial_z(\gamma E^z + \gamma v B^y) \\
&= \gamma \operatorname{div} \boldsymbol{E} - \gamma v \left((\operatorname{curl} B)^x - \partial_t E^x\right) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
(\operatorname{curl}' E')^{x'} + \partial_t (B')^{x'} &= \partial_y(\gamma E^z + \gamma v B^y) - \partial_z(\gamma E^y - \gamma v B^z) + \gamma \partial_t B^x + \gamma v \partial_x B^x \\
&= \gamma v \operatorname{div} \boldsymbol{B} + \gamma\left(\partial_t B^x + (\operatorname{curl} \boldsymbol{E})^x\right) \\
&= 0
\end{aligned}
$$

The other components follow from very similar calculations.

c) i) We can calculate

$$\partial^a F_{ab} = \begin{pmatrix} -\operatorname{div}\boldsymbol{E} \\ -\partial_t E^x + (\operatorname{curl}\boldsymbol{B})^x \\ -\partial_t E^y + (\operatorname{curl}\boldsymbol{B})^y \\ -\partial_t E^z + (\operatorname{curl}\boldsymbol{B})^z \end{pmatrix} \qquad \partial^a * F_{ab} = \begin{pmatrix} -\operatorname{div}\boldsymbol{B} \\ -\partial_t B^x - (\operatorname{curl}\boldsymbol{E})^x \\ -\partial_t B^y - (\operatorname{curl}\boldsymbol{E})^y \\ -\partial_t B^z - (\operatorname{curl}\boldsymbol{E})^z \end{pmatrix}$$

ii) We can calculate, for example,

$$\begin{aligned}(E')^{x'} = F'_{01} &= (\Lambda^{-1})_0{}^a(\Lambda^{-1})_1{}^b F_{ab} \\ &= \gamma^2 F_{01} + \gamma^2 v^2 F_{10} \\ &= E^x\end{aligned}$$

$$\begin{aligned}(E')^{y'} = (F'_{02} &= (\Lambda^{-1})_0{}^a(\Lambda^{-1})_2{}^b F_{ab} \\ &= \gamma F_{02} + \gamma v F_{12} \\ &= \gamma E^y - \gamma v B^z\end{aligned}$$

and the other components follow similarly.

iii) We could try defining, for example, the spacetime vector $E = (0, \mathbf{E})$, but then under a Lorentz boost we would have

$$E' = \begin{pmatrix} -\gamma v E^x \\ \gamma E^x \\ E^y \\ E^z \end{pmatrix}$$

If, instead, we thought of E as a covector then we would have

$$E' = \begin{pmatrix} \gamma v E^x \\ \gamma E^x \\ E^y \\ E^z \end{pmatrix}$$

but neither of these are the correct transformation laws.

More generally, Lorentz transformations *mix* the components of the electric and magnetic fields, so they can't be considered separate objects. Instead, the object we should consider is the electromagnetic field tensor, which transforms like the product of two covectors.

5. Calculating

$$\dot{y}^i = \frac{\partial y^i}{\partial x^j}\dot{x}^j + \frac{\partial y^i}{\partial t}$$

$$\ddot{y}^i = \frac{\partial^2 y^i}{\partial x^j \partial x^k}\dot{x}^j\dot{x}^k + 2\frac{\partial^2 y^i}{\partial x^j \partial t}\dot{x}^j + \frac{\partial^2 y^i}{\partial t^2}$$

Setting $F = m\ddot{y}$ we obtain the expression for the force.

Most forces in nature are proportional to some intrinsic property of the object on which they are acting – for example, the electromagnetic force is proportional to the *charge* of a particle on which it acts. *Gravity* – like a fictitious force – is proportional to mass.