Problem sheet 2

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All questions use natural units, where Newton's constant G and the speed of light c are both equal to 1.

Questions marked with a star * are optional extension questions which go beyond the scope of the course. They will not be discussed in class unless all other questions have already been covered. You are advised to only attempt these questions if you have already completed the other questions on the sheet.

Questions/comments/correction to Joseph.Keir@maths.ox.ac.uk

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1. *The sphere as a manifold*

Show explicitly that the sphere $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ is a smooth manifold. You will need to use at least two coordinate patches!

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2. *Practice with tensors*

Explain why each of the following equations is *not* a well-formed tensor equation.

a) $X^\mu = Y_\mu$

b) $X^\mu + Y^\nu = 0$

c) $T^{\mu\nu} - Y^\mu = 0$

d) $X^\mu = 1$

e) $X^{\mu\nu\rho} Y_{\mu\nu} Z^\nu = 0$

f) $\phi = (X^\mu Y_\mu)(A^\mu B_\mu)$

g) $R^{\mu\mu} = 3$

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3. *Coordinate induced vectors, covectors and bases*

Suppose that U is some coordinate patch on a manifold, with ϕ_U the associated coordinate chart, and let $p \in U$. Define the curves γ_a, through the point p, where

$$\tilde{\gamma}_a := \phi_U \circ \gamma_a$$

is the curve on $\mathbb{R}^n$ such that $x^b =$ constant, for all $b \neq a$. Parametrise this curve by x^a.

a) Show that the *coordinate induced vector fields* $\{\partial_a\}$, where ∂_a is the tangent vector to the curve γ_a at the point p, span the tangent space at p, $T_p(\mathcal{M})$.

b) Likewise, show that the *coordinate induced covector fields* $\{\mathrm{d}x^a\}$ span the cotangent space at p, $T_p^*(\mathcal{M})$.

c) Now suppose that we choose to work in new coordinates, wherein only the last coordinate differs from the coordinate previously used. That is, we use coordinates $\{y^a\}$, where

$$y^0 = x^0 \quad , \quad y^1 = x^1 \quad \ldots \quad y^{n-1} = x^{n-1} \quad , \quad y^n = f(x^0, \ldots, x^n)$$

i) Writing ∂'_a for the vector constructed from the coordinates y^a in the same way that the vector ∂_a was constructed from the coordinates x^a, and assuming that $\frac{\partial f}{\partial x^n}\Big|_p \neq 0$, show that

$$\partial'_a = \partial_a - \frac{\partial f}{\partial x^a}\bigg|_{x^b, b\neq a} \left(\frac{\partial f}{\partial x^n}\right)^{-1} \partial_n \qquad a = 0, \ldots, n-1$$

$$\partial'_n = \left(\frac{\partial f}{\partial x^n}\right)^{-1} \partial_n$$

where everything is evaluated at the point p.

ii) Show also that

$$\mathrm{d}y^a = \mathrm{d}x^a \qquad a = 0, \ldots, n-1$$
$$\mathrm{d}y^n = \frac{\partial f}{\partial x^a}\mathrm{d}x^a$$

This helps us to make sense of the names *covariant* and *contravariant*: under a change of coordinates, covectors like $\mathrm{d}y^a$ vary '*with*' the coordinates (hence *co*-variant), while vectors vary '*against*' the coordinates (hence *contra*-variant).

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4. *The Hessian*

Let f be some smooth function on a manifold. Then, given vector fields X and Y, we define

$$H_f(X, Y) := \frac{1}{2}X(Y(f)) + \frac{1}{2}Y(X(f))$$

Let p be a point in the manifold.

a) Under what conditions on the function f does H_f define a $(0, 2)$ tensor at p?

b) In the case where H_f does define a tensor, show that, in local coordinates x^a, this tensor is the *Hessian* of f, i.e.

$$(H_f)_{ab} = \frac{\partial^2 f}{\partial x^a \partial x^b}$$

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5. *Outer/tensor products*

Let $T^{\mu\nu}$ be a $(2, 0)$ tensor at some point p in a manifold $\mathcal{M}$.

a) Show that it is always possible to write

$$T^{\mu\nu} = \sum_{(A)} X^\mu_{(A)} Y^\nu_{(A)}$$

for some vector fields $X_{(A)}$ and $Y_{(A)}$.

Bonus: show that, if T is a tensor on an n dimensional manifold, then one actually only needs to sum over at most n pairs of vector fields. In other words,

$$T^{\mu\nu} = \sum_{(A)=1}^{n} X^{\mu}_{(A)} Y^{\nu}_{(A)}$$

b) Some $(0, 2)$ tensors can be written in terms of a *single* pair of vectors, i.e.

$$T^{\mu\nu} = X^{\mu} Y^{\nu}$$

Working in some local coordinates x^a, write down a condition on the components T^{ab} which is sufficient to guarantee that $T^{\mu\nu}$ *cannot* be written in terms of a single pair of vector fields in this way.

c) Give a *coordinate-independent* condition on the tensor $T^{\mu\nu}$ that is both necessary and sufficient for there to exist vector fields X and Y such that $T^{\mu\nu} = X^{\mu} Y^{\nu}$.

Hint: it may help to examine the properties of the linear map

$$\begin{aligned} \tilde{T} : T^*_p(\mathcal{M}) &\to T^*_p(\mathcal{M}) \\ \eta &\mapsto T(\eta, \cdot) \end{aligned}$$

where $T(\eta, \cdot)$ is the vector such that the action of an arbitrary covector μ is given by $\mu\left(T(\eta, \cdot)\right) = T(\eta, \mu)$.

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6. *The Christoffel symbols of Minkowski space in various coordinate systems*

Recall the expression for the Christoffel symbols of the Levi-Civita connection:

$$\Gamma^{a}_{bc} = \frac{1}{2}(g^{-1})^{ad}\left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\right)$$

a) Show that, in the usual rectangular coordinates, the Christoffel symbols of the Minkowski metric vanish.

b) In cylindrical coordinates (t, ρ, ϕ, z) show that

i) the Minkowski metric is given by

$$g = -\mathrm{d}t^2 + \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\phi^2 + \mathrm{d}z^2$$

ii) the nonzero Christoffel symbols are

$$\begin{aligned} \Gamma^{\rho}_{\phi\phi} &= -\rho \\ \Gamma^{\phi}_{\rho\phi} &= \rho^{-1} \\ \Gamma^{\phi}_{\phi\rho} &= \rho^{-1} \end{aligned}$$

c) In spherical polar coordinates (t, r, θ, ϕ), show that

i) the Minkowski metric is given by

$$g = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)$$

ii) the nonzero Christoffel symbols are

$$\begin{aligned}
\Gamma^r_{\theta\theta} &= -r \\
\Gamma^r_{\phi\phi} &= -r\sin^2\theta \\
\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} &= r^{-1} \\
\Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta \\
\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} &= r^{-1} \\
\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} &= \cot\theta
\end{aligned}$$

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7. *Covariant derivatives and geodesics*

Suppose that the vector Y is parallel transported along an affinely parametrised geodesic with tangent vector X.

a) Let s be the affine parameter along the geodesic. Show that the components of Y satisfy the linear differential equation

$$\frac{\mathrm{d}}{\mathrm{d}s}Y^a + \left(\Gamma^a_{bc}X^b\right)Y^c = 0$$

b) Show that $g(X,X)$, $g(X,Y)$ and $g(Y,Y)$ are all constant along the geodesic.

c) Suppose the vector Z is also parallel transported along the geodesic. Show that $g(Y,Z)$ is constant along the geodesic.

d) Now let K be a covector field satisfying the equation

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0$$

such a covector field is called a *Killing covector field*. Show that $K(X)$ is constant along the geodesic (even if K is not parallel transported!).

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8. *The exterior derivative of a covector field*

a) Suppose that we are working in local coordinates x^a. Recall that, given two vector fields X, Y, their commutator is defined as the vector field with components

$$[X,Y]^a = X^b\frac{\partial Y^a}{\partial x^b} - Y^b\frac{\partial X^a}{\partial x^b}$$

Suppose that we change to different coordinates $y^{a'} = y^{a'}(x^a)$. Show that the components of $[X,Y]$ in this new coordinate system are given by

$$[X,Y]^{a'} = X^{b'}\frac{\partial Y^{a'}}{\partial y^{b'}} - Y^{b'}\frac{\partial X^{a'}}{\partial y^{b'}}$$

Hence this expression transforms as a vector field, and consequently the vector field $[X,Y]$ can be defined *without* reference to any specific local coordinates.

b) Let η be a covector field. The exterior derivative of η is defined as

$$\mathrm{d}\eta(X,Y) = X\left(\eta(Y)\right) - Y\left(\eta(X)\right) - \eta\left([X,Y]\right)$$

where X and Y are any two vector fields.

i) Show that $\mathrm{d}\eta$ is a $(0,2)$ tensor field.

ii) In terms of local coordinates, show that the components of $\mathrm{d}\eta$ are given by

$$(\mathrm{d}\eta)_{ab} = \partial_a \eta_b - \partial_b \eta_a$$

so these kinds of tensor fields are antisymmetric. Totally antisymmetric $(0,2)$ tensor fields are called *two-forms* (even if they cannot be written as $\mathrm{d}\eta$ for some covector η). Covector fields themselves are sometimes called *one-forms*, and higher rank, totally antisymmetric covariant tensor fields are called *p-forms* (where p is the rank).

iii) This definition has not made use of a connection or even a metric. However, if a metric is available to us, show also that

$$(\mathrm{d}\eta)_{ab} = \nabla_a \eta_b - \nabla_b \eta_a$$

iv) Suppose that $\eta = \mathrm{d}f$ for some smooth function f. Show that, in this case, $\mathrm{d}\eta = 0$.

c) Recall that covariant tensors are defined as multilinear maps from vectors to the real line. The *restriction* of a covariant tensor to some submanifold (that is, to some hypersurface or lower dimensional surface, like a line) is simply the restriction of this multilinear map to vector fields which are tangent to this surface – that is, to vectors which are the tangent vectors to curves which lie entirely within the submanifold. In practice it is often simpler to use coordinates, then the restriction of the covector

$$\eta = \eta_0 \mathrm{d}x^0 + \eta_1 \mathrm{d}x^1$$

to the surface $x^0 =$ constant is simply

$$\eta\big|_{x^0=\text{constant}} = \eta_1 \mathrm{d}x^1$$

Suppose that we want to integrate the two-form $\mathrm{d}\eta$ over some two-dimensional surface Σ. We work in local coordinates x^a, which we chose so that the surface Σ is given by

$$\Sigma = \Big\{ p \in \mathcal{M} \,:\, x^0(p) \in [0,1] \;,\; x^1(p) \in [0,1] \;,\; x^a(p) = 0 \text{ for } a > 1 \Big\}$$

In this case, the integral of $\mathrm{d}\eta$ over Σ is defined as

$$\int_\Sigma \mathrm{d}\eta = \int_\Sigma \mathrm{d}\eta\big|_\Sigma = \int_\Sigma \left(\mathrm{d}\eta\big|_\Sigma\right)_{01} \mathrm{d}x^0 \mathrm{d}x^1$$

Note that this definition requires a choice of orientation for the surface Σ: we are saying that (x^0, x^1) is a 'right-handed' coordinate system on Σ.

ii) Show that, if we had made the other choice, so that (x^1, x^0) (in that order) was defined as 'right-handed', then the integral would acquire a minus sign.

ii) Suppose that we change coordinates on Σ, from (x^0, x^1) to (y^0, y^1). Show that, in these new coordinates, the integrand is

$$|J| \left(\mathrm{d}\eta\big|_\Sigma\right)_{01}$$

where $|J|$ is the determinant of the Jacobian matrix associated with the change of coordinates, and $\left(\mathrm{d}\eta\big|_\Sigma\right)_{01}$ is the $(0,1)$-component of the tensor $\mathrm{d}\eta\big|_\Sigma$ with respect to the (x^0, x^1) coordinate system. *(Note the relationship between this formula and the usual expression for the change of an integrand when changing variables in multivariate calculus!)*

iii) Prove the following form of Stokes' theorem:

$$\int_\Sigma \mathrm{d}\eta\big|_\Sigma = \int_{\partial\Sigma} \eta\big|_{\partial\Sigma}$$

which can be written even more compactly as

$$\int_\Sigma \mathrm{d}\eta = \int_{\partial\Sigma} \eta$$

(All of this can be generalised to higher rank forms – and higher dimensional surfaces – as well as to surfaces which are not "square". This forms the basis of the theory of integration on manifolds.)