C7.5: General Relativity I
Problem Sheet 3

Please submit solutions in the collection boxes in the basement of the Mathematical Institute no later than Tuesday 20th November, 4pm. Corrections to `ashmore@maths.ox.ac.uk`.

1. Independent components of the Riemann tensor

The Riemann tensor $R_{ab}{}^{c}{}_{d}$ associated to a torsion-free ($\Gamma^{a}{}_{[bc]} = 0$) covariant derivative ∇_a obeys the following algebraic identities:

1. $R_{ab}{}^{c}{}_{d} = -R_{ba}{}^{c}{}_{d}$
2. $R_{[ab}{}^{c}{}_{d]} = 0$

What is the additional algebraic identity obeyed when ∇_a is the Levi-Civita covariant derivative compatible with a g_{ab}? In this case, show that the Riemann tensor in four dimensions has 20 independent components. How many independent components does it have it D dimensions?

2. Riemann tensor in two dimensions

1. Show that in two dimensions, the Riemann tensor must be of the form

$$R_{abcd} = \frac{1}{2} R (g_{ac} g_{bd} - g_{ad} g_{bc}),$$

   where R is the Ricci scalar. *Hint: express the Ricci scalar R in terms of the one independent component of the Riemann tensor, say R_{1212}.*

2. Show that Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$ vanishes automatically in two dimensions.

3. Consider the two-dimensional de Sitter metric

$$\mathrm{d}s^2 = -\mathrm{d}u^2 + \cosh^2 u \, \mathrm{d}\varphi^2,$$

   where $-\infty < u < \infty$ and $0 \leq \varphi < 2\pi$. Using your results from question 5 of problem sheet 2, compute the one independent component of the Riemann tensor, say $R_{u\varphi u\varphi}$.

4. Compute the Ricci scalar R for this two-dimensional de Sitter metric and verify that $R_{abcd} = g_{ac} g_{bd} - g_{ad} g_{bc}$.

3. Newtonian limit of geodesic deviation

The relative acceleration of a one-parameter family of geodesics $x_t^a(\lambda) = x^a(\lambda, t)$ is determined by the geodesic deviation equation

$$\frac{D^2 N^d}{D\tau^2} = R_{ab}{}^d{}_c V^a N^b V^c,$$

where

$$V^a = \frac{\mathrm{d}x^a}{\mathrm{d}\lambda}, \qquad N^b = \frac{\mathrm{d}x^b}{\mathrm{d}t},$$

and $D/D\tau = V^a \nabla_a$ is the covariant derivative in the direction of V^a.

In the Newtonian limit, show that this reduces to

$$\frac{\partial^2 N^i}{\partial t^2} = -\sum_j \frac{\partial^2 \phi}{\partial x^i \partial x^j} N^j,$$

where ϕ is the gravitational potential.

4. Schwarzschild solution

Check that the Schwarzschild solution satisfies the vacuum field equations $R_{ab} = 0$. You should do this carefully in your own time; it is not necessary to hand this calculation in.

5. Circular orbits

The Lagrangian for affinely parametrised geodesics in the Schwarzschild solution is

$$\mathcal{L} = -(1 - 2M/r)\dot{t}^2 + (1 - 2M/r)^{-1}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2).$$

Show that

$$E = (1 - 2M/r)\dot{t}, \qquad J = r^2 \sin^2\theta\,\dot{\phi},$$

are conserved. What symmetries do they correspond to? Explain why the Lagrangian itself is conserved and why $\mathcal{L} = -1$ for time-like geodesics parametrised by proper time. Show that, without loss of generality, you can always restrict to time-like geodesics lying in the equatorial plane, $\theta = \pi/2$.

Now consider a circular orbit in the equatorial plane at coordinate distance R. Show that this is a time-like geodesic if $R > 3M$ and find the conserved quantities (E, J).

6. Orbital periods

The Schwarzschild metric describing spacetime outside a spherical mass M is given by

$$\mathrm{d}s^2 = -(1 - 2M/r)\mathrm{d}t^2 + (1 - 2M/r)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2).$$

1. Using conserved quantities and the change of variables $r = 1/u$, find the radius R at which a photon can follow a circular orbit. *Hint: use $\mathcal{L} = 0$ for a null geodesic to find an*

expression for $\dot{r}^2$, then rewrite in terms of $du/d\phi$.

2. What would a fixed observer at $r = R$ measure as the time period of the photon's orbit?

3. If this fixed observer flashes a signal each time the photon passes, at what interval would a stationary observer at $r \to \infty$ measure these flashes?