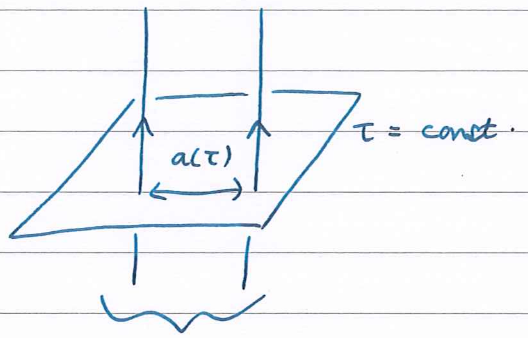
15.1 - Review

Homogeneity + Isotropy:

$$ds^2 = -d\tau^2 + a(\tau)^2 \begin{cases} d\psi^2 + \sin^2\psi\,(d\theta^2 + \sin^2\theta\, d\phi^2) \\ d\psi^2 + \psi^2(d\theta^2 + \sin^2\theta\, d\phi^2) \\ d\psi^2 + \sinh^2\psi\,(d\theta^2 + \sin^2\theta\, d\phi^2) \end{cases}$$



Isotropic / comoving observers with proper time τ and $u^a = (1, \underline{0})$

$a(\tau)$ is scale factor, fixed by Einstein equations.

15.2 - Curvature

Focus on <u>flat</u> spatial geometry $(k=0)$

$$ds^2 = -d\tau^2 + a^2\, dx^i dx^i$$

1) Compute Γ from $\mathcal{L}$

$$\mathcal{L} = -\dot{\tau}^2 + a^2 \dot{x}^i \dot{x}^i \quad , \quad \dot{\tau} = \frac{d\tau}{d\lambda} \quad , \quad a' = \frac{da}{d\tau}$$

$$\Rightarrow \quad \Gamma^\tau{}_{ii} = a a' \quad , \quad \Gamma^i{}_{i\tau} = \frac{a'}{a}$$

NB: $\Gamma^{\tau}{}_{ii}$ and $\Gamma^{i}{}_{i\tau}$ equal for any i due to isotropy.

2) Isotropy $\Rightarrow$ R_{aa} (no sum) are possibly non-zero.

Find $$R_{\tau\tau} = -\frac{3a''}{a}$$

$$R_{ii} = aa'' + 2(a')^2$$

3) $$R = g^{ab} R_{ab}$$

$$= -R_{\tau\tau} + \frac{1}{a^2}\sum_{i=1}^{3} R_{ii}$$

$$= \frac{3a''}{a} + \frac{1}{a^2} 3\left(aa'' + 2(a')^2\right)$$

$$= 6\left(\frac{a''}{a} + \frac{(a')^2}{a^2} \right)$$

Recall $$G_{ab} \equiv R_{ab} - \tfrac{1}{2} R g_{ab}$$

$$\Rightarrow \quad G_{\tau\tau} = R_{\tau\tau} + \frac{1}{2} R$$

$$= \frac{3(a')^2}{a^2}$$

$$\Rightarrow \quad G_{ii} = -2aa'' - (a')^2$$

15.3 - Matter

Need to specify matter content of universe

In general one finds

$$G_{00} = 3\left(\frac{(a')^2}{a^2} + \frac{k}{a^2}\right)$$

$$G_{ij} = -\left(\frac{2a''}{a} + \frac{(a')^2}{a^2} + \frac{k}{a^2}\right) h_{ij}$$

$a^2 \, ds^2_{(3)}$

Given $G_{ab} = 8\pi T_{ab}$, we must have

$$T_{00} = \rho(\tau)$$

$$T_{ij} = p(\tau) \, h_{ij}$$

Written in terms of the 4-velocity of a comoving observer u_a

$$T_{ab} = (p+\rho)\, u_a u_b + p\, g_{ab}$$

i.e. perfect fluid at rest w.r.t. comoving coordinates.

Supplement with equation of state $p = w\rho$

$S = S(V^w E)$

e.g. dust : $w = 0$ $(p = 0)$

radiation : $w = 1/3$

cosmological constant : $w = -1$

NB : $$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$$

$$\Rightarrow G_{ab} = 8\pi \left(T_{ab} - \frac{\Lambda}{8\pi} g_{ab} \right)$$

$$\Rightarrow T^{\Lambda}{}_{ab} = -\frac{\Lambda}{8\pi} g_{ab}$$

$$\Rightarrow T^{\Lambda}_{00} \equiv \rho_\Lambda = \frac{\Lambda}{8\pi} = -p_\Lambda$$

(can also have $w = w(\tau)$, e.g. inflation).

15.4 - Einstein equations

Focus on $k = 0$, $T_{00} = \rho$

$$T_{ii} = a^2 p$$

$$G_{ab} = 8\pi T_{ab}$$

$\hookrightarrow$ 1) $$3\left(\frac{a'}{a}\right)^2 = 8\pi \rho$$

2) $$-2aa'' - (a')^2 = 8\pi a^2 p$$

1) + 2) $\Rightarrow$ $$\frac{a''}{a} = -\frac{4\pi}{3}(\rho + 3p)$$

Conservation of T_{ab}

$$D_a T^{ab} = 0 \Rightarrow \rho' + 3(\rho + p)\frac{a'}{a} = 0$$
(τ component)

For <u>any</u> geometry

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi\rho}{3} - \frac{k}{a^2}$$

$$\frac{a''}{a} = -\frac{4\pi}{3}(\rho + 3p)$$

$$\rho' + 3(\rho + p)\frac{a'}{a} = 0$$

(not independent, any 2 imply the 3rd)
c.f. Bianchi

<u>15.5 - Examples</u>

a) <u>Dust</u> : pressureless gas $p = 0$ (e.g. galaxies)

Continuity eq$^{\underline{n}}$: $\rho' + 3\rho\frac{a'}{a} = 0$

$$\Rightarrow \frac{\rho'}{\rho} + \frac{3a'}{a} = 0$$

$$\Rightarrow \rho(a) \propto a^{-3}$$

Note: - ρ is rest mass per unit volume measured by isotropic observers

- Rest mass conserved

- Spatial volumes scale as a^3

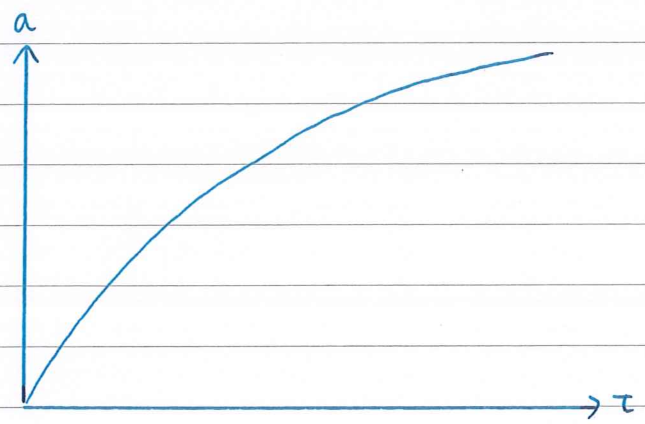
- expect $\rho \sim a^{-3}$

For $k = 0$

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi}{3}\rho = c\, a^{-3}$$

$$\Rightarrow \quad a' = \sqrt{c}\, a^{-1/2}$$

$$\Rightarrow \quad a(\tau) \propto \tau^{2/3}$$



v) <u>Radiation</u> filled universe, $p = \frac{1}{3}\rho$

Continuity: $\rho' + 4\rho \frac{a'}{a} = 0$

$$\Rightarrow \quad \rho(a) \propto a^{-4}$$

- a^{-3} from volume scaling
- a^{-1} from redshift of photons

For $k = 0$, $a(\tau) \propto \tau^{1/2}$

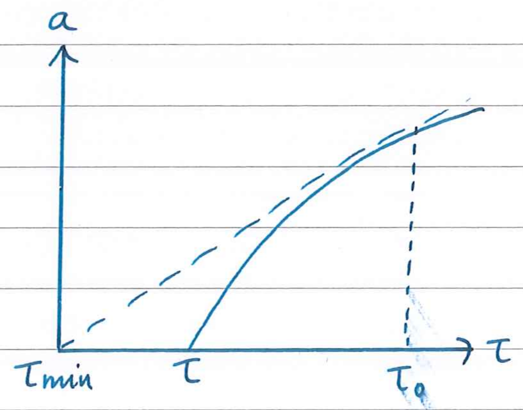
15.6 – General conclusions

From $\frac{a''}{a} = -\frac{4\pi}{3}(\rho + 3p)$ ($a > 0$ by definition)

if $\rho > 0$, $p > 0$, $\Rightarrow a'' < 0$

i.e. universe cannot be static!

Given $a' > 0$ today (expansion) and $a'' < 0$ (slowing expansion), we must have $a = 0$ at some point τ in the past



Given $a' > 0$, fate of universe determined by $k = \pm 1, 0$.

From continuity equation, if $p > 0$, ρ decreases at least as fast as a^{-3}

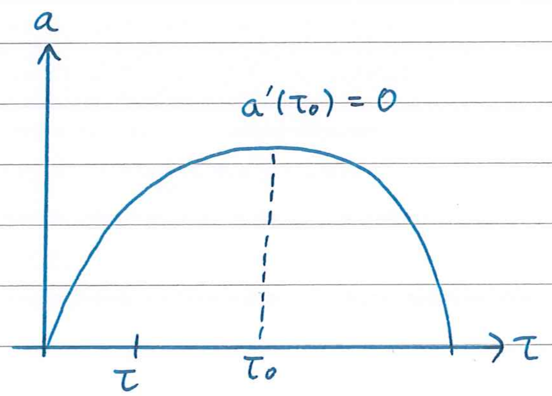
From $\left(\frac{a'}{a}\right)^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2}$

$\underline{k = 0, -1}$: If $a' > 0$ today, RHS $\neq 0$ so $a' > 0$ for all future

"Universe expands forever"

$\underline{k = +1}$: If $a' > 0$ today, ρ decreases more rapidly than a^{-2} as a increases.

RHS $= 0$ for some τ, then $a' < 0$. for future.



Universe with compact spatial geometry (S^3) exists for finite time!

NB: Λ contributes $p < 0$ so these conclusions change depending on Λ!