14.1 - Cosmology

Cosmology is large-scale description of universe.

- Cannot do easy "cosmological experiments" so there are many things we don't understand
- What happened at (or before) Big Bang?
- Is universe spatially finite or infinite?
- Will universe continue expanding?
- What is dark matter?
- Is dark energy causing acceleration? What is dark energy?

Build models using <u>observations</u>

- Impossible to write down models for all stars etc.
- Average over galactic scales to get an approximate model.

1) At large scales, non-gravitational interactions can be ignored (short range or screened).

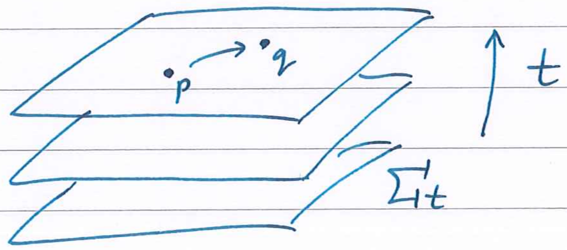
~~2)~~

2) No privileged position in the universe - "Copernican Principle". Universe looks the same everywhere, i.e. space is homogeneous.

3) Universe looks the same in all directions. Rotational symmetry, i.e. space is isotropic.

14.2 - Homogeneity

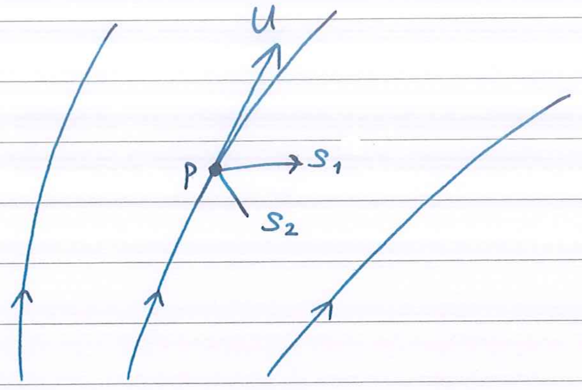
Spacetime foliated by 1-parameter family of spacelike hypersurfaces



For any 2 points $p, q \in \Sigma_t$, there is a transformation $p \mapsto q$ that preserves the metric - an "isometry".

14.3 - Isotropy

There is a set of distinguished observers (timelike geodesics) filling spacetime such that



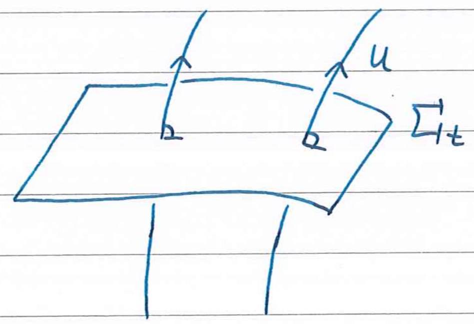
$$g(u, s_1) = g(u, s_2) = 0$$

for any point p and a pair of spacelike vectors orthogonal to u, there exists an isometry of the metric that rotates $s_1 \mapsto s_2$.

Known as "isotropic observers".

13.4 - Metric

Homog. + Iso. $\Rightarrow$ Σ_t orthogonal to u



$$\Rightarrow \quad g_{ab} = u_a u_b + h_{ab}$$

h_{ab} → metric on Σ_t

Introduce coordinates : $x^a = (\tau, x^i)$

(co-moving)
τ = proper time of isotropic observers

(homog $\Rightarrow$ isotropic observers measure same $\Delta\tau$ between hypersurfaces)

In these coordinates

$$ds^2 = -d\tau^2 + h_{ij}\,dx^i\,dx^j$$

Let $R_{ij}{}^k{}_l$ be the Riemann tensor on Σ_τ built from h_{ij}

Claim: 1) Isotropy $\Rightarrow$ $R_{ijkl} = \frac{R}{6}(g_{ik}g_{jl} - g_{jk}g_{il})$

2) Homog. $\Rightarrow$ R is constant on Σ_τ (depends on τ only)

(Proof of 1) at end of notes)

A space with R_{ijkl} of this form and R = const is a space of constant curvature.

Taken together, Σ_τ must be a maximally symmetric space. There are 3 of these

- $R > 0$: 3-sphere S^3 with round metric

$$ds^2_{(3)} = d\psi^2 + \sin^2\psi\,(d\theta^2 + \sin^2\theta\,d\phi^2)$$

- $\underline{R=0}$: Flat $\mathbb{R}^3$ with Euclidean metric

$$ds^2_{(3)} = d\psi^2 + \psi^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$

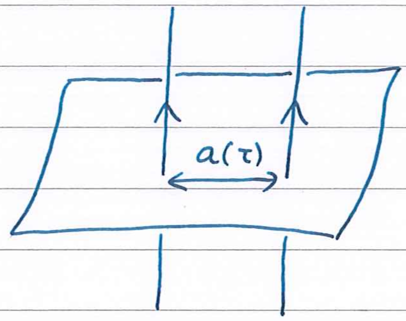
- $\underline{R<0}$: Hyperbolic space H^3

$$ds^2_{(3)} = d\psi^2 + \sinh^2\psi \, (d\theta^2 + \sin^2\theta \, d\phi^2)$$

The 4d metric is then <u>Friedmann - Robertson - Walker</u>

$$ds^2 = -d\tau^2 + a^2(\tau) \, ds^2_{(3)}$$

The <u>scale factor</u> $a(\tau)$ gives separation of nearly isotropic observers



$a(\tau) > 0$ by definition.

To determine $a(\tau)$, solve Einstein equation given

- Choice of $R>0$, $R=0$ or $R<0$.
- Choice of matter content.
- Choice of cosmological constant Λ (to come).

Comments

- Can also take

$$ds_{(3)}^2 = \frac{dr^2}{1-kr^2} + \underbrace{r^2 d\Omega^2}_{S^2}$$

- Previous metric has $k=1$ for S^3, etc.

- In this metric, can use rescaling of $a(\tau)$ to set $k = \pm 1, 0$.

- Or use scaling to set $a(\tau_0) = 1$ where τ_0 = "today".

- At constant τ, $ds^2 = a(\tau)^2 ds_{(3)}^2$

  - For $k=1$, $a(\tau)$ gives size (radius) of universe.

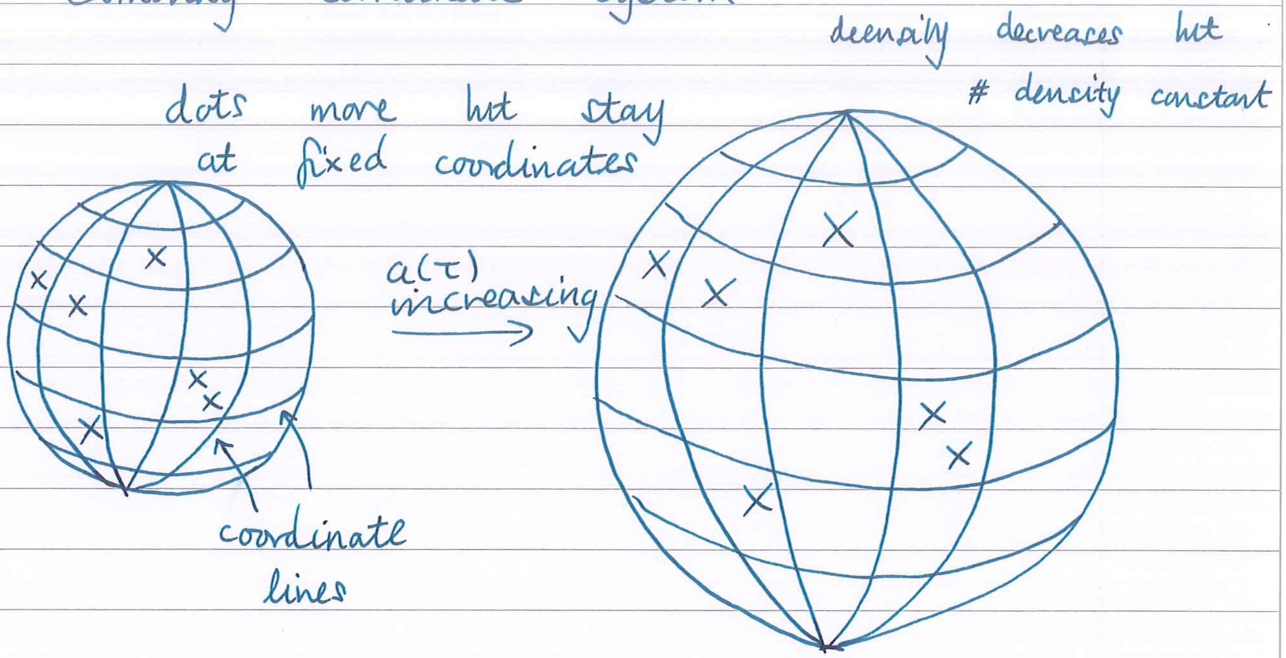
  - For $k=-1$, universe is infinite but $a(\tau)$ sets scale of curvature

$$R^{(3)}(\tau) = \frac{1}{a^2(\tau)} R^{(3)}$$

  - For $k=0$, $\mathbb{R}^3$ infinite.

NB: $k=1$ case gives option of unbounded finite universe (S^3 is boundary of ball in 4d).

e.g. Comoving coordinate system



One can check that x^i = constant curves are geodesics. (or ψ = constant)

Co-moving observers at ψ = constant follow geodesics.

Their proper time is just the τ that appears in the metric.

<u>Proof of $R_{ijkl} = \frac{R}{6}(g\ldots +\quad)$</u>

From $R_{ij}{}^{kl} = -R_{ji}{}^{kl} = -R_{ij}{}^{lk}$, we have a linear map on the vector space of ~~rank~~ $(0,2)$ antisymmetric tensors at $p \in \Sigma_\tau$, W.

$$R : W \to W ,$$

$$: A_{ij} \to R_{ij}{}^{kl} A_{kl}$$

Since $R_{ij}{}^{kl} = + R^{kl}{}_{ij}$, the map is self-adjoint with respect to the +ve definite inner product on W

$$\langle -, - \rangle : W \times W \to \mathbb{R}$$

$$: (A_{ij}, B_{ij}) \mapsto h^{ik} h^{jl} A_{ij} B_{kl}$$

W is finite dimensional, so has an orthonormal basis of eigenvectors of $R : W \to W$.

If the eigenvalues were distinct, we could construct distinguished vectors which would violate isotropy.

Thus all eigenvalues must be equal

$\Rightarrow \quad R = $ identity map.

$$\Rightarrow \quad R = k \, \mathbb{1}$$

i.e. $\quad R_{ij}{}^{kl} = k(\delta_i{}^k \delta_j{}^l - \delta_j{}^k \delta_i{}^l)$

Contracting to get $\quad R = 6k$

Homogeneity implies R cannot depend on x^i, so it is constant on Σ_t.