13.1 - Singularities

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

What happens at $r = 2M, 0$?

Singularities can be artifact of coordinates

Compute scalar invariants:

- $R \equiv g^{ab} R_{ab} = 0$ (as solves Einstein)
- $R_{abcd} R^{abcd} = \dfrac{12M^2}{r^6}$

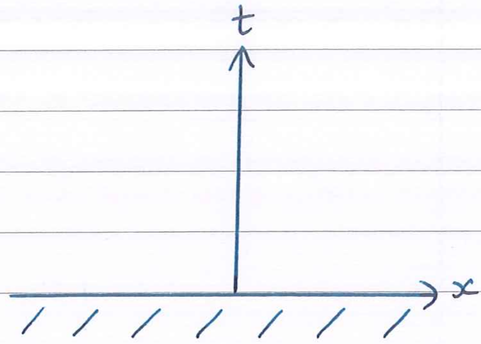
$r = 0$ is genuine singularity in geometry

Nothing happens to R^2 at $r = 2M$. Singularity in metric components can be removed by a coordinate transformation

$r = 2M$ plays important physical role, known as "event horizon".

13.2 - Toy Example I

$$ds^2 = -\frac{dt^2}{t^4} + dx^2$$



$$0 < t < \infty \,, \quad -\infty < x < \infty$$

Singularity in metric components at $t = 0$

Change coordinates : $t' = t^{-1}$

$$dt' = -\frac{dt}{t^2}$$

$$\Rightarrow \quad ds^2 = -dt'^2 + dx^2$$

Metric on half of flat $\mathbb{R}^2$ with $0 < t' < \infty$

Singularity at $t = 0 \equiv t' \to \infty$ region in flat $\mathbb{R}^2$.

<u>Definition</u>: Spacetime is <u>geodesically complete</u> if all geodesics can be extended to arbitrarily ~~large~~ values of affine parameter.

(t', x) coordinates show geodesics can extend out to $t' \to \infty$ $(t = 0)$ without issue

- Spacetime not geodesically complete as $t' = 0$ is limit.

Can extend spacetime so it is geodesically complete.

- In (t, x), extend past $t \to \infty$
- In (t', x), just include $t' \leq 0$ to get full $\mathbb{R}^2$.

13.3 – Toy Example II

Rindler spacetime

$$ds^2 = -x^2 dt^2 + dx^2$$

$$-\infty < t < \infty$$
$$0 < x < \infty$$

Metric components singular at $x = 0$

$$g_{ab} = \begin{pmatrix} -x^2 & \\ & 1 \end{pmatrix} \qquad g^{ab} = \begin{pmatrix} -1/x^2 & \\ & 0 \end{pmatrix}$$

- Geodesics terminate with finite length at $x = 0$
- Curvature invariants not singular at $x = 0$ ($R_{abcd} = 0$!)

Introduce coordinates using null geodesics

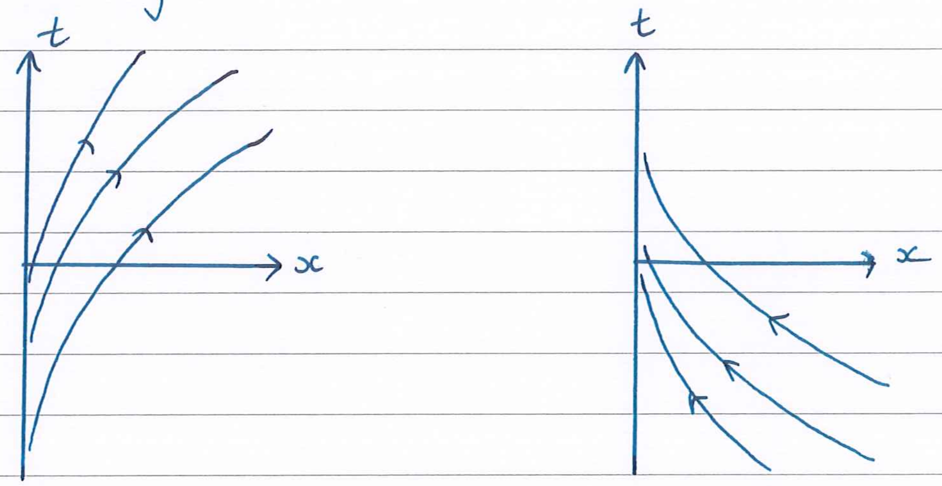
$$L = -x^2\dot{t}^2 + \dot{x}^2 = 0$$

$$\Rightarrow \left(\frac{dt}{dx}\right)^2 = \frac{1}{x^2}$$

$$\Rightarrow \quad t = \pm \log x + \text{const.}$$

$+$: outgoing

$-$: incoming



Define "null coordinates" (u, v)

$$u = t - \log x$$

$$v = t + \log x$$

Incoming null geodesic $\Rightarrow$ $v = \text{const.}$

Outgoing null geodesic $\Rightarrow$ $u = \text{const.}$

Compute metric in (u, v) coordinates

$$du = dt - \frac{dx}{x}$$

$$dv = dt + \frac{dx}{x}$$

$$\Rightarrow \quad du\,dv = dt^2 - \frac{dx^2}{x^2}$$

$$\Rightarrow \quad ds^2 = -e^{v-u}\,du\,dv$$

where $-\infty < u < \infty$ and $-\infty < v < \infty$ corresponds to $x > 0$

Note, since $x^2 = e^{v-u}$, $x = 0$ corresponds to $v = -\infty$ or $u = +\infty$.

Is the space geodesically complete? Look near $x = 0$.

Let's compute the affine parameter λ along null geodesics.

$$\frac{\partial}{\partial t} L = 0 \quad \Rightarrow \quad E = x^2 \dot{t} = \text{const.}$$

$$= x^2 \frac{dt}{d\lambda}$$

$$\Rightarrow \quad d\lambda = \frac{1}{E} x^2\,dt$$

$$= \frac{1}{2E} e^{v-u} (du + dv)$$

<u>Outgoing</u> : $u = u_0$ (constant)

$$\Rightarrow \quad \lambda = \frac{1}{2E} \int e^{v-u_0} \, dv$$

$$= C + \frac{e^{-u_0}}{2E} e^{v}$$

$$\equiv a + b\lambda'$$

i.e. e^{v} is affine parameter

<u>Incoming</u> : $v = v_0$

$$\Rightarrow \quad \lambda = \frac{1}{2E} \int e^{-u+v_0} \, du$$

$$= C - \frac{e^{v_0}}{2E} e^{-u}$$

i.e. $-e^{-u}$ is affine parameter

Can we follow geodesics for <u>any</u> length?

i.e. do affine parameters take values in <u>all</u> of $\mathbb{R}$?

<u>No</u>! $\lambda_{in} = e^{v}$, $\lambda_{out} = -e^{-u}$

$\lambda_{in} \geq 0$, $\lambda_{out} \leq 0$

Idea : change coordinates so that metric has no singularity at $x = 0$, then just extend range of coordinates to give new, extended spacetime.

Let $U = -e^{-u}$, $V = e^{v}$, $U \in (-\infty, 0)$
$V \in (0, \infty)$

$$\Rightarrow \quad ds^2 = -dU\,dV$$

No singularity at $U = V = 0$ $(x = 0)$

- Extend to $U \in (-\infty, \infty)$
  $V \in (-\infty, \infty)$

- This extended spacetime is geodecically complete (cannot access new region with old (u, v) coords.)

Finally, set $T = \frac{1}{2}(U+V)$

$$X = \frac{1}{2}(V-U)$$

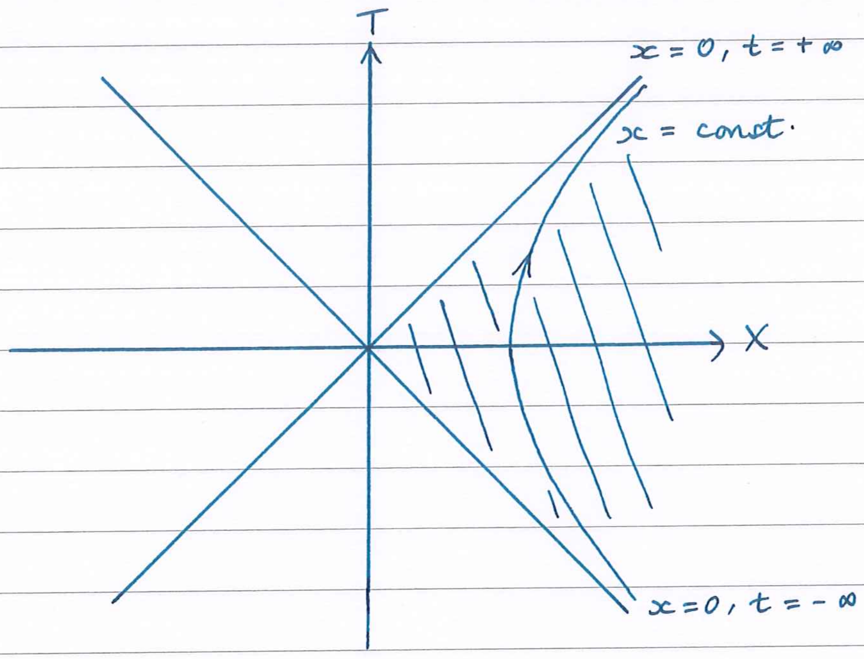
$$\Rightarrow \quad ds^2 = -dT^2 + dX^2$$

$$T \in (-\infty, \infty), \quad X \in (-\infty, \infty)$$

Simply $\mathbb{R}^2$ with flat metric!

If there were a curvature singularity at $x = 0$, no reason to extend spacetime!

- Original spacetime is region $X > |T|$
- Null geodesics are <u>straight lines</u>.
- Curve $x = \text{const.}$ is uniformly acc. observer in (T, X) coordinates



13.4 – Back to Schwarzschild

In $r > 2M$ region

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

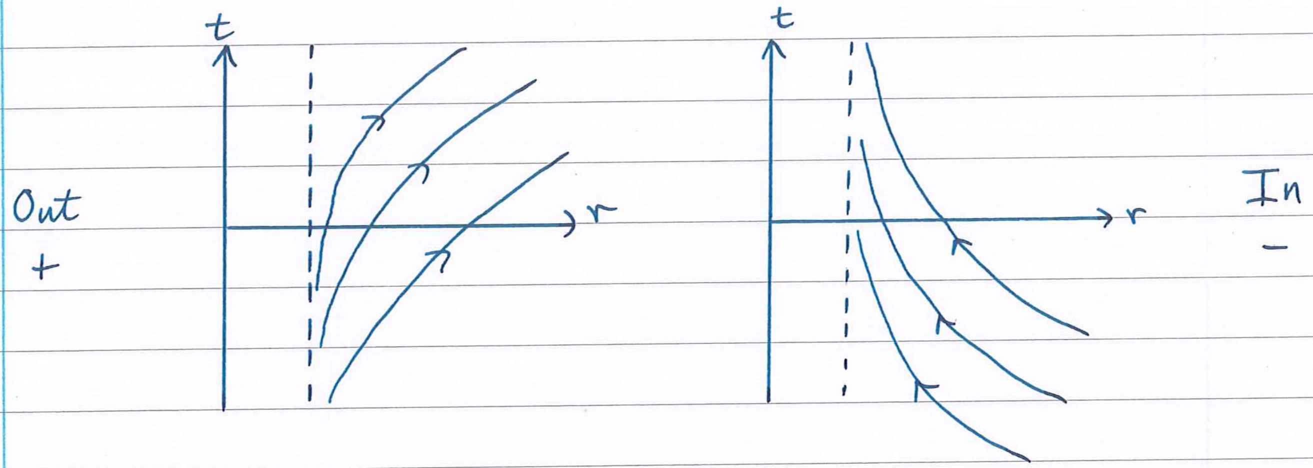
Apply same method: look at null geodesics, find coordinates well-behaved at $r = 2M$, extend coordinates.

Null geodesics: $\mathcal{L} = 0 = -\left(1-\frac{2M}{r}\right)\dot{t}^2 + \left(1-\frac{2M}{r}\right)\dot{r}^2$

$$\Rightarrow \left(\frac{dt}{dr}\right)^2 = \left(\frac{1}{1-\frac{2M}{r}}\right)^2$$

$$\Rightarrow \quad t = \pm r_* + \text{const.}$$

where $\quad r_* = r + 2M \log\left(\frac{r}{2M} - 1\right)$



Use null coordinates in $r > 2M$

$$u = t - r_*$$

$$v = t + r_*$$

Incoming: $v = v_0$

Outgoing: $u = u_0$

Metric: $$ds^2 = -\left(1 - \frac{2M}{r}\right) du\, dv$$

$$= -\frac{2M}{r}\, e^{-r/2M}\, e^{(v-u)/4M}\, du\, dv$$

with $u \in (-\infty, \infty)$, $v \in (-\infty, \infty)$

$r \to 2M$ corresponds to $u \to +\infty$ or $v \to -\infty$

This is not geodesically complete : radial null geodesics meet $r = 2M$ at <u>finite</u> affine parameter

c.f. $\left(1 - \frac{2M}{r}\right)\dot{t} = \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}$ for null

$$\Rightarrow \quad dt = \pm \frac{dr}{1 - \frac{2M}{r}}$$

but $E = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda}$

$$\Rightarrow \quad d\lambda = \pm \frac{1}{E}\, dr$$ so <u>r is affine</u>.

<u>Define</u> : $U = -e^{-u/4M} \in (-\infty, 0)$

$V = e^{v/4M} \in (0, \infty)$

$$\Rightarrow \quad ds^2 = -\frac{32M^3}{r} e^{-r/2M}\, dU\, dV$$

Metric is <u>non-singular</u> at $r = 2M$ ($U = 0$ or $V = 0$) so we extend to

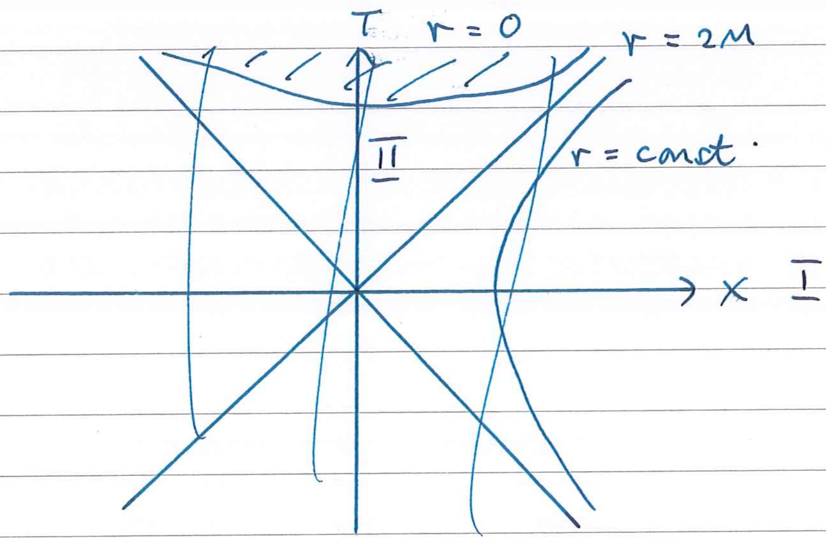
$U \in (-\infty, \infty)$, $V \in (-\infty, \infty)$

Finally : $T = \frac{1}{2}(U+V)$

$X = \frac{1}{2}(V-U)$

- $r > 0$ requires $T^2 - X^2 < 1$
- Metric : $ds^2 = \frac{32m^2}{r} e^{-r/2m}(-dT^2 + dX^2)$
- Null geodesics : incoming $T + X = \text{const.}$

  outgoing $T - X = \text{const}$
- $r > 2M$ region is $X > |T|$
- Now nothing strange happens at $r = 2M$. Can follow a particle down to $r = 2M$ using (t, r), then follow across horizon with with (U, V), then convert back to (t, r) inside horizon.



Consider observer falling across horizon

For $r < 2M$

$$ds^2 = + \underbrace{\left(\frac{2M}{r} - 1\right)}_{>0} dt^2 \cancel{+} - \underbrace{\left(\frac{2M}{r} - 1\right)}_{>0}^{-1} dr^2$$

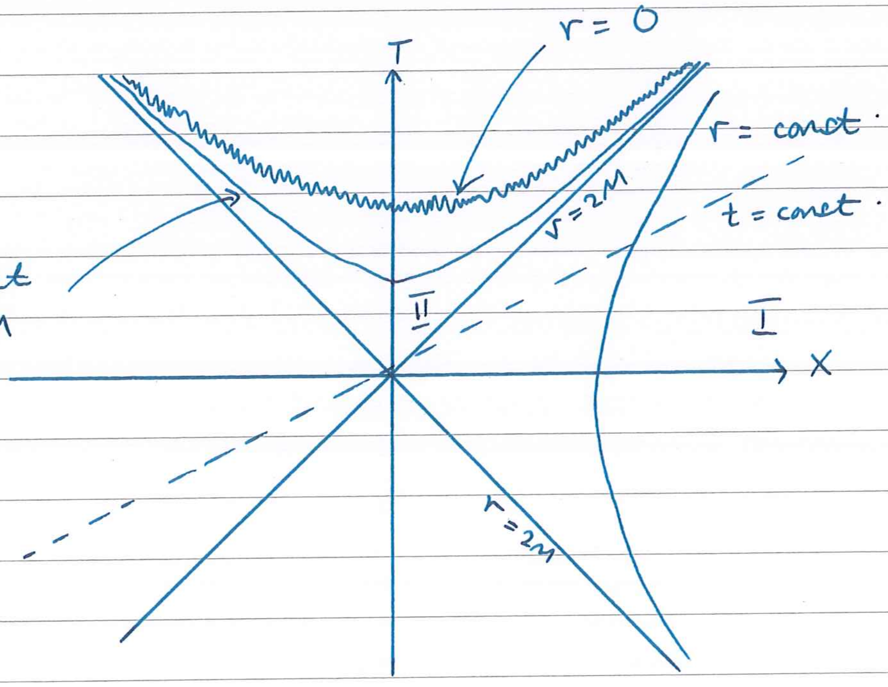
t is now a spacelike coordinate.

r is timelike coordinate.

~~For a timelike geodesic~~

- r decreases as particles move to the future

- $r = 0$ (where $R_{abcd} R^{abcd}$ diverges) is a point in time. It cannot be avoided.



- Particle can never escape from $r < 2M$
- Will reach $r = 0$ in finite proper time
- Signals sent from II always hit $r = 0$

"<u>Black Hole</u>"