11.1 - Geodesics for Schwarzschild

Lagrangian for affinely parametrised geodesics

$$\mathcal{L} = g_{ab}\,\dot{x}^a \dot{x}^b \qquad (\text{set } G = 1)$$

$$= -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2$$

$$+\, r^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right)$$

Conserved quantities

1) $\mathcal{L}$ independent of t $\Rightarrow$ $\dfrac{d}{d\lambda}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{t}}\right) = 0$

$$E := \left(1 - \frac{2M}{r}\right)\dot{t}$$

2) $\mathcal{L}$ independent of ϕ :

$$J = r^2 \sin^2\theta\,\dot{\phi}^2$$

3) $\mathcal{L}$ itself is conserved

$$-K = \mathcal{L} = \begin{cases} -1 & \text{TL geodesic} \\ 0 & \text{null geodesic} \end{cases}$$

In TL case, choose proper time as parameter.

Spherical symmetry $\Rightarrow (J_x, J_y, J_z)$ conserved

- J corresponds to ϕ rotations
- Other components are messy

Simplification: Can always rotate coordinates so particles initial motion is in $\theta = \frac{\pi}{2}$ plane, WLOG.

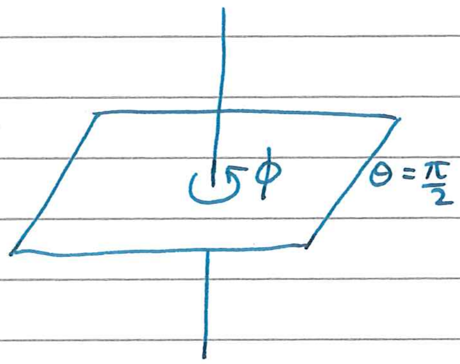
Particle motion stays in this plane

Proof: θ eq^{n} $\quad \frac{d}{d\lambda}(r^2 \dot{\theta}) = r^2 \sin\theta \cos\theta \dot{\phi}^2$

$$\Rightarrow \quad \ddot{\theta} + \frac{2\dot{r}\dot{\theta}}{r} - \sin\theta \cos\theta \dot{\phi}^2 = 0 \quad ⊛$$

Use symmetry to set

$\theta(0) = \frac{\pi}{2}, \qquad \dot{\theta}(0) = 0$



Then ⊛ implies $\ddot{\theta}(0) = 0$

$$\Rightarrow \theta(s) = \frac{\pi}{2}$$

Then the only non-zero angular momentum is J.

Restrict to $\theta = \frac{\pi}{2}$, $\dot{\theta} = 0$

$$E = \left(1 - \frac{2M}{r}\right)\dot{t}$$

$$J = r^2 \dot{\phi}$$

$$\mathcal{L} = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2$$

$$= -K$$

Combine (eliminate $\dot{t}$ and $\dot{\phi}$ from $\mathcal{L}$)

$$-K = \frac{\dot{r}^2 - E^2}{1 - \frac{2M}{r}} + \frac{J^2}{r^2}$$

$$\Rightarrow \quad E^2 = \dot{r}^2 + \left(K + \frac{J^2}{r^2}\right)\left(1 - \frac{2M}{r}\right)$$

$$\Rightarrow \quad \frac{E^2 - K}{\cancel{m}\,2} = \frac{1}{2}\dot{r}^2 + V(r)$$

$$V(r) = \frac{-KM}{r} + \frac{J^2}{2r^2} - \frac{mJ^2}{r^3}$$

"Motion of massive particle in 1d with effective potential $V(r)$"

11.2 - Timelike Geodesics

TL $\Rightarrow$ $K = 1$

$$\Rightarrow \quad \frac{E^2 - 1}{2} = \frac{1}{2}\dot{r}^2 + V(r)$$

$$V(r) = \underbrace{\frac{-M}{r}}_{\text{Newtonian gravitational potential}} + \underbrace{\frac{J^2}{2r^2}}_{\text{Centrifugal barrier}} - \underbrace{\frac{mJ^2}{r^3}}_{\text{GR correction}}$$

Compare to energy of non-relativistic particle in $1/r$ potential.

$$\mathcal{L} = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\phi}^2 + \frac{mM}{r}$$

$$\mathcal{H} = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\phi}^2 - \frac{M}{r}$$

$$= \frac{1}{2}\dot{r}^2 + \underbrace{\frac{J^2}{2r^2} - \frac{M}{r}}_{V_N(r)}$$

i.e. $$V(r) = V_N(r) - \frac{mJ^2}{r^3}$$

Properties of $V(r)$

- $V(r) \to -\frac{M}{r}$ as $r \to \infty$
- $V(r) \to \frac{-mJ^2}{R^3}$ as $r \to 0$

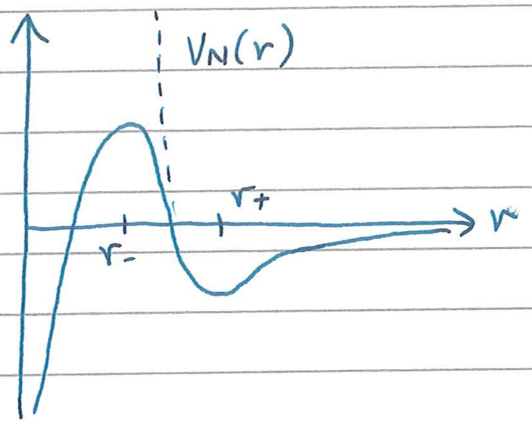
Extrema : $$\frac{\partial V}{\partial r} = \frac{M}{r^2} - \frac{J^2}{r^3} + \frac{3mJ^2}{r^4}$$

$V' = 0$ at $$r_\pm = \frac{J^2}{2M}\left(1 \pm \sqrt{1 - 12\left(\frac{M}{J}\right)^2}\right)$$

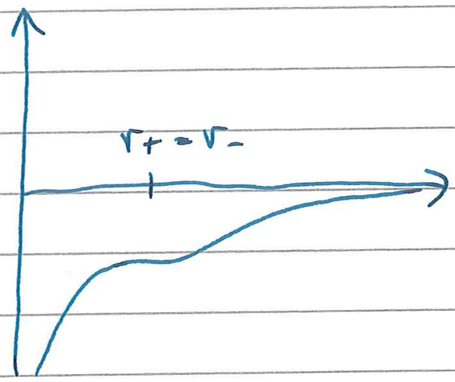
At $J/M = \sqrt{12}$, extrema collide to leave inflection point.

$J/M > \sqrt{12}$

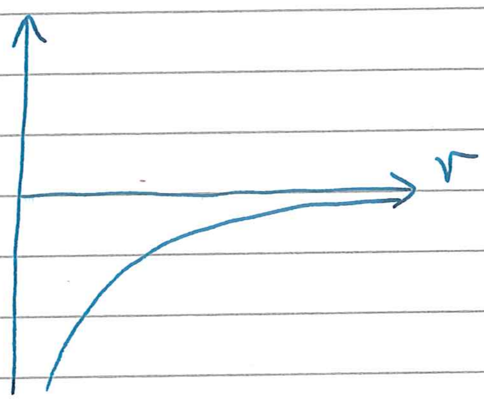
r_+ is minimum.



$J/M = \sqrt{12}$



$J/M < \sqrt{12}$



For this choice, the GR correction completely overcomes the angular momentum barrier ~~for~~ from the centrifugal term.

"Gravity in GR stronger than Newtonian"

11.3 - Types of Orbit

Recall : $\frac{E^2-1}{2} = \frac{1}{2}\dot{r}^2 + V(r)$

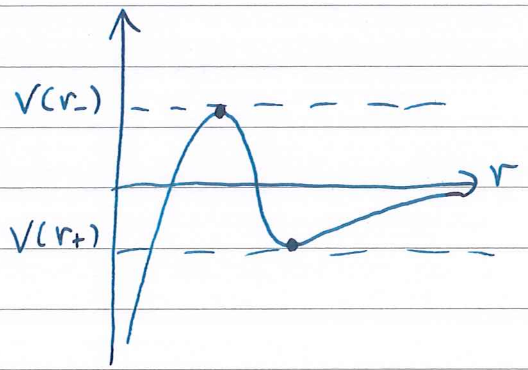
$$\Rightarrow \ddot{r} = -\frac{\partial V}{\partial r} \quad \text{(can also get from } r \text{ eq}^{\underline{n}}\text{)}$$

a) Circular orbit

$\dot{r} = \ddot{r} = 0$

$\ddot{r} = 0 \Rightarrow \frac{\partial V}{\partial r} = 0$

which is true for $r = r_\pm$



$r_+ > r_-$

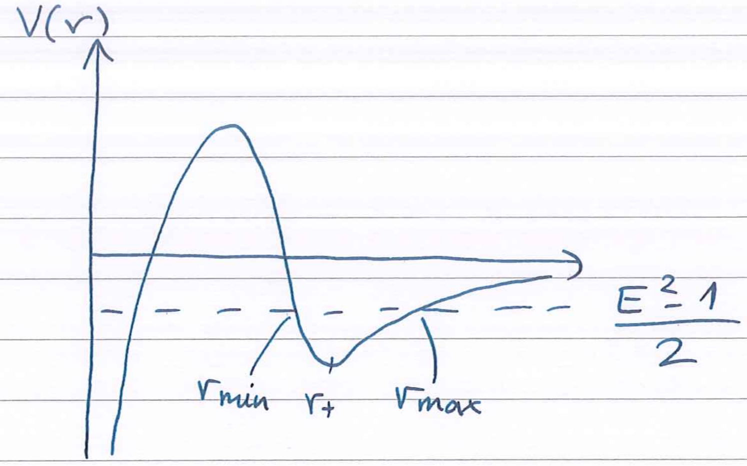
- r_- is unstable orbit

- r_+ is stable orbit

The closest (marginally) stable orbit is for $J/M = \sqrt{12}$

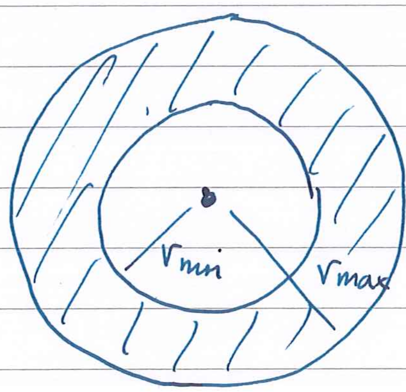
$$r_\pm = \frac{J^2}{2M} = 6M$$

b) Bound Orbits



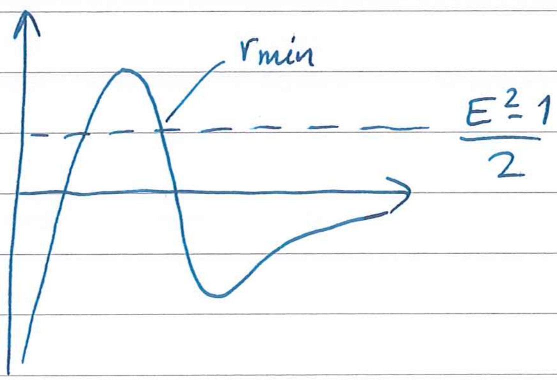
$$V(r_+) < \frac{E^2-1}{2} < 0$$

where $r > r_-$

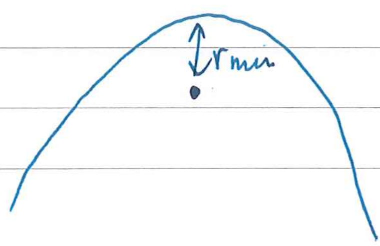


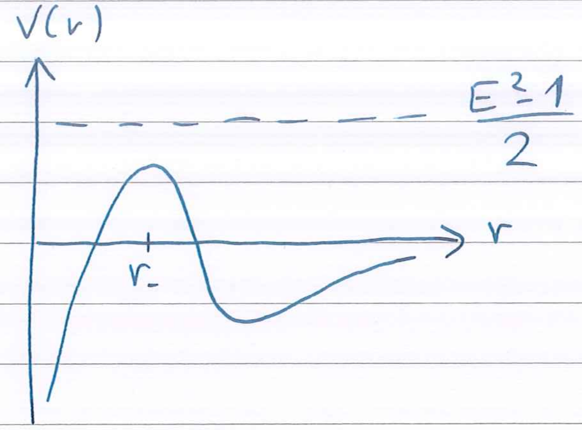
Orbit confined to shaded region.

c) Unbound orbits

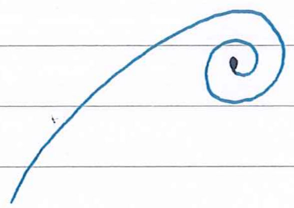


$$0 < \frac{E^2-1}{2} < V(r_-)$$





$$\frac{E^2-1}{2} > V(r_-)$$



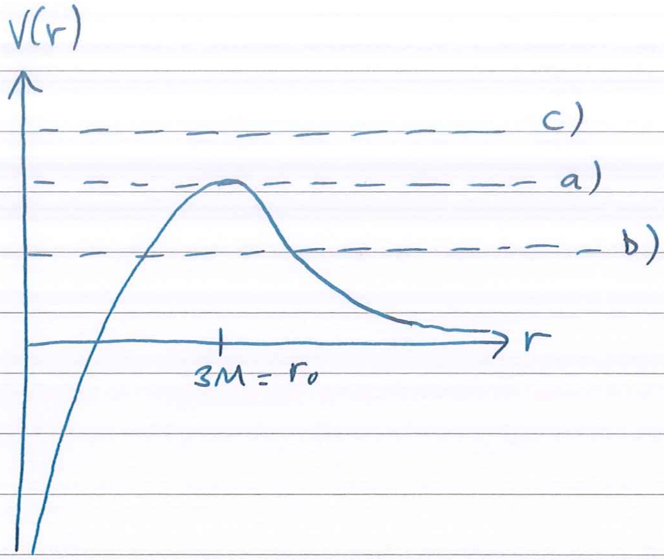
Such an orbit is not possible in Newtonian theory as angular momentum barrier is infinitely high.

11.4 - Null geodesics

$$k = 0 \Rightarrow \frac{E^2}{2} = \frac{1}{2}\dot{r}^2 + V(r)$$

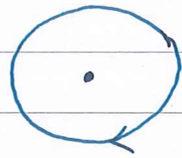
$$V(r) = \frac{J^2}{2r^2} - \frac{MJ^2}{r^3}$$

No Newtonian potential term as have mass~~ive~~less particles



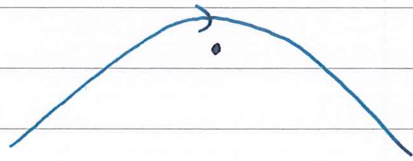
a) Unstable circular orbit

$r_0 = 3M$, $\frac{E^2}{2} = V(r_0)$



b) Unbound orbit

$0 < \frac{E^2}{2} < V(r_0)$



c) Unbound orbit

$\frac{E^2}{2} > V(r_0)$