10 - Schwarzschild Solution

10.1 - Vacuum equations

Absence of mass / energy : $T_{ab} = 0$

$$\Rightarrow \quad R_{ab} - \frac{1}{2} g_{ab} R = 0 \quad \big\downarrow g^{ab}$$

$$\Rightarrow \quad R - 2R = 0$$

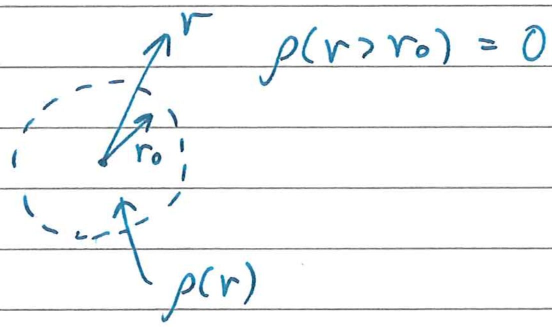
$$\Rightarrow \quad R_{ab} = 0$$ "vacuum equations"

10.2 - Schwarzschild solution.

Gravitational field outside spherical distribution of matter.

Birkhoff's theorem: ~~any~~ there is a unique static, spherically symmetric solution of $R_{ab} = 0$.

(spherically $\Rightarrow$ static + asymp. flat $\Rightarrow$ Schwarzschild)



Metric outside :

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2MG}{r}\right)^{-1} dr^2$$

$$+ r^2 \underbrace{\left(d\theta^2 + \sin^2\theta \, d\phi^2 \right)}_{\substack{\text{slice at fixed } (t,r) \\ \text{has } S^2 \text{ geometry}}}$$

r : radial coordinate

t : time coordinate

$R_s = 2GM$ is Schwarzschild radius

Restoring units : $R_s = \frac{2GM}{c^2}$

- Sun : $R_s \simeq 1$ km
- Earth : $R_s \simeq 1$ cm

Astrophysics cares about $r > R_s$ region.

Newtonian limit: 1) Weak field for $r \gg R_s$.

2) $\frac{dt}{dt} \gg \frac{dr}{d\tau}$ non-relativistic.

$$g_{tt} = -\left(1 - \frac{2GM}{r}\right)$$

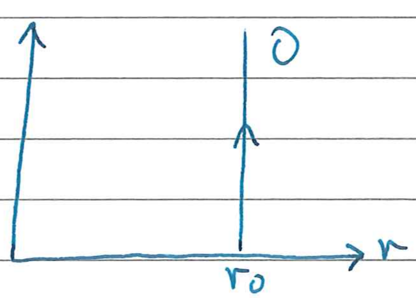
$$\Rightarrow h_{00} = \frac{2GM}{r}$$

Compare with $\phi = -\frac{1}{2} h_{00} = -\frac{GM}{r}$

from last lecture

10.3 - Stationary observers

Stationary: $r = r_0 = \text{const}$.



Question: how is proper time of O related to t?

t is "coordinate time".

Since $dr = 0$ for $\mathcal{O}$ (and $d\theta = d\phi = 0$)

$$ds^2 = -d\tau^2 = -\left(1 - \frac{2GM}{r_0}\right) dt^2$$

$$\Rightarrow \quad \Delta\tau = \left(1 - \frac{2GM}{r_0}\right)^{1/2} \Delta t$$

As $r_0 \gg R_s$, $\Delta\tau \rightarrow \Delta t$

Coordinate time t is the proper time of a stationary observer at spatial infinity $r \rightarrow \infty$.

Alternatively: $\mathcal{O}$ is stationary

$$u^a = \frac{dx^a}{d\tau} = \left(\frac{dt}{d\tau}, 0, 0, 0\right)$$

$$u_a u^a = -1$$

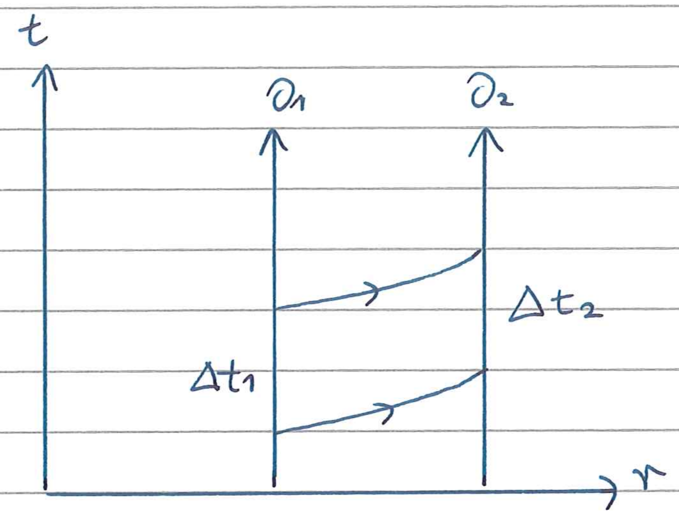
$$= g_{00}(u^0)^2$$

$$= -\left(1 - \frac{2GM}{r_0}\right)\left(\frac{dt}{d\tau}\right)^2$$

$$\Rightarrow \quad \frac{dt}{d\tau} = \frac{1}{\left(1 - \frac{2GM}{r_0}\right)^{1/2}}$$

10.4 - Gravitational Redshift

Consider 2 stationary observers O_1 and O_2



- O_1 emits signals separated by Δt_1.
- O_2 receives signals separated by Δt_2

Spacetime is static (metric does not depend on t)

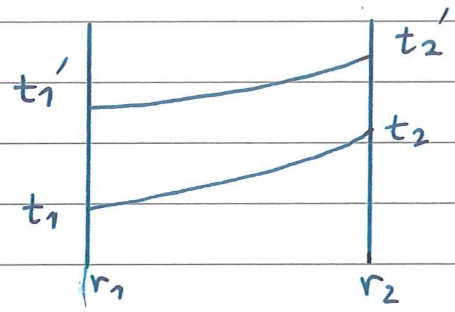
$\Rightarrow$ $\Delta t_1 = \Delta t_2 = \Delta t$ "coordinate time agrees"

Explicitly: light travels on null geodesic, a curve that obeys

$$ds^2 = 0$$

$$\Rightarrow \left(1 - \frac{2MG}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 = 0$$

$$\Rightarrow \quad \frac{dt}{dr} = + \frac{1}{\left(1 - \frac{2GM}{r}\right)^{1/2}} \quad \text{(+ as } t \text{ increases with } r\text{, outgoing)}$$



$$\int_{t_1}^{t_2} dt = \int_{r_1}^{r_2} \frac{1}{\left(1 - \frac{2GM}{r}\right)^{1/2}} = \int_{t_1'}^{t_2'} dt$$

$$\Rightarrow \quad t_2 - t_1 = t_2' - t_1'$$

$$\Rightarrow \quad \Delta t_1 = \Delta t_2$$

How much proper time passes for O_1 and O_2?

Stationary $\Rightarrow dr = d\theta = d\phi = 0$

$$\Rightarrow \quad -d\tau^2 = -\left(1 - \frac{2GM}{r}\right) dt^2$$

$$\Rightarrow \quad d\tau = \left(1 - \frac{2GM}{r}\right)^{1/2} dt$$

$$\Rightarrow \quad \Delta \tau_i = \left(1 - \frac{2GM}{r_i}\right)^{1/2} \Delta t$$

So $$\frac{\Delta\tau_2}{\Delta\tau_1} = \left(\frac{1 - \frac{2GM}{r_2}}{1 - \frac{2GM}{r_1}} \right)^{1/2}$$

As $r_2 > r_1$, $\Delta\tau_2 > \Delta\tau_1$

Less time passes for O_1 than O_2

O_1's clock "runs slow"

Clocks run slow in gravitational field / curved spacetime.

"Gravitational Time Dilation".

Example: Near surface of Earth

$$r_1 = R_E$$

$$r_2 = R_E + h \quad , \quad h \ll R_E$$

$$\frac{\Delta\tau_2}{\Delta\tau_1} \simeq 1 - \frac{GM}{R_E + h} + \frac{GM}{R_E} \quad (\text{expand } 1/2)$$

$$\simeq 1 + \frac{GM}{R_E^2} h$$

$$= 1 + gh \quad \text{"shift in photon } \omega \text{"}$$

Consider two different limits

1) $r_2 \to \infty$, $\mathcal{O}_2$ far away from source

$$\frac{\Delta\tau_2}{\Delta\tau_1} \to \frac{1}{\left(1 - \frac{2GM}{r_1}\right)^{1/2}} \quad \circledast$$

$\Delta\tau_2 = \Delta t$

Can be used to operationally define r.

Stationary observer $\mathcal{O}_2$ at ∞ sends signals to $\mathcal{O}_1$ separated by $\Delta\tau_2$.

$\mathcal{O}_1$ sends message back with how long between signals ($\Delta\tau_1$)

$\mathcal{O}_2$ then defines $\mathcal{O}_1$ to be at r_1 given by $\circledast$

2) $r_1 \to 2GM$

$$\frac{\Delta\tau_2}{\Delta\tau_1} \to \infty \; !$$

$\mathcal{O}_2$ observes $\mathcal{O}_1$ to slow down and eventually "freeze" as they pass the Schwarzschild radius.

<u>NB</u>: For stars, etc, solution only valid for $r > R_E$ and $R_E \gg 2GM_E$, so do not see this!

↑ radius of body