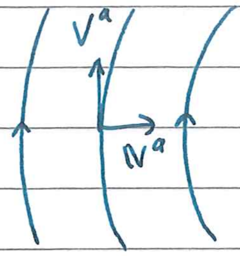
9.1 - Review

Geodesic deviation : $\frac{D^2 N^a}{D\lambda^2} = R_{bc}{}^{a}{}_{d} V^b N^c V^d$



Einstein equations : $R_{ab} - \frac{1}{2} g_{ab} = \lambda T_{ab}$

Plan : Determine λ by going to Newtonian limit and comparing to Newton's 2nd law.

9.2 - Newtonian limit : curvature

Assume existence of appnx. global frame with :

1) $g_{ab} = \eta_{ab} + \varepsilon h_{ab} + \mathcal{O}(\varepsilon^2)$

2) $\partial_t g_{ab} = 0 + \mathcal{O}(\varepsilon)$

Gravity is : 1) Weak.

2) Slowly varying.

View h_{ab} as perturbation around flat space.

- To $\mathcal{O}(\varepsilon)$, raise and lower with η_{ab}, η^{ab}.

Now want to compute curvature

1) Inverse metric : $g_{ab} = \eta_{ab} + \varepsilon h_{ab}$

$$\Rightarrow \quad g^{ab} = \eta^{ab} - \varepsilon h^{ab}$$

so that $g_{ab} g^{bc} = \delta_a^{\ c} + \mathcal{O}(\varepsilon^2)$

2) Christoffel symbols

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad} \left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc} \right)$$

$$= \frac{1}{2} \varepsilon \, \eta^{ad} \left(\partial_b h_{cd} + \partial_c h_{bd} - \partial_d h_{bc} \right) + \mathcal{O}(\varepsilon^2)$$

3) Riemann tensor : since $\Gamma \sim \mathcal{O}(\varepsilon)$, only $\partial \Gamma$ is $\mathcal{O}(\varepsilon)$ (ignore Γ^2)

$$R_{ab}{}^{c}{}_{d} = \partial_a \Gamma^c{}_{bd} - \partial_b \Gamma^c{}_{ad} + \mathcal{O}(\varepsilon^2)$$

$$= \frac{1}{2} \varepsilon \Big[\eta^{ce} \left(\partial_a \partial_b h_{de} + \partial_a \partial_d h_{be} - \partial_a \partial_e h_{bd} \right)$$

$$\updownarrow 0$$

$$- \eta^{ce} \left(\partial_b \partial_a h_{de} + \partial_b \partial_d h_{ae} - \partial_b \partial_e h_{ad} \right) \Big]$$

$$R_{abcd} = \frac{\varepsilon}{2}\Big(\partial_a \partial_d h_{bc} - \partial_a \partial_c h_{bd} + \partial_b \partial_c h_{ad} - \partial_b \partial_d h_{ac} \Big) + \mathcal{O}(\varepsilon^2)$$

Comment: symmetries are manifest.

4) Ricci tensor:

$$\begin{aligned} R_{ac} &= g^{bd} R_{abcd} \\ &= \eta^{bd} R_{abcd} + \mathcal{O}(\varepsilon^2) \\ &= \frac{\varepsilon}{2}\Big(\partial_a \partial_b h^b{}_c - \partial_a \partial_c h + \partial_b \partial_c h_a{}^b - \partial_b \partial^b h_{ac} \Big) \\ &= \frac{\varepsilon}{2}\Big(2\, \partial^b \partial_{(a} h_{c)b} - \partial_b \partial^b h_{ac} - \partial_a \partial_c h \Big) \end{aligned}$$

where $h := \eta^{ab} h_{ab} = h^a{}_a = h_a{}^a$

5) Einstein tensor: introduce "trace-free" perturbation (or trace reversed)

$$\bar{h}_{ab} = h_{ab} - \tfrac{1}{2} \eta_{ab} h \qquad (\mathrm{tr}\, \eta_{ab} = 2)$$

Then: $\bar{h} = h - 2h = -h \;\Rightarrow\; h_{ab} = \bar{h}_{ab} - \tfrac{1}{2} \eta_{ab} \bar{h}$

$$G_{ac} := R_{ac} - \frac{1}{2} g_{ac} R$$

$$= \frac{\varepsilon}{2}\left(2\, \partial^b \partial_{(a} \bar{h}_{c)b} - \partial_b \partial^b \bar{h}_{ac} - \eta_{ac} \partial^b \partial^d \bar{h}_{bd} \right)$$

6) Gauge invariance

$g_{ab} = \eta_{ab} + \varepsilon h_{ab} + O(\varepsilon^2)$ does not fully specify the coordinate system.

- There can be other coordinates in which $g = \eta + h + \dots$ but h will be different.
- Decomposition of metric into background + perturbation is not unique.
- Different h's can give same curvature, i.e. same physical spacetime to $O(\varepsilon^2)$

Gauge transformation:

$$x^a \mapsto x^a + \xi^a(x)$$

$$h_{ab} \mapsto h_{ab} - \partial_a \xi_b - \partial_b \xi_a$$

Comment : can check $\delta R_{abcd} = 0$ to $O(\xi^2)$
so h'_{ab} describes same geometry.

Want to match to Newton

- Pick "harmonic" gauge (nice coordinates)
- Also known as Einstein / Hilbert / de Donder gauge.

$$\partial^a \bar{h}_{ab} = 0$$

- Weak field limit of

$$g^{ab} \Gamma^c{}_{ab} = 0 \quad \Leftrightarrow \quad D_a \partial^a x^b = \Box x^b = 0$$

$\uparrow$ not a vector

(Cartesian coordinates in SR solve this too!)

$$\Rightarrow G_{ac} = -\frac{\varepsilon}{2} \partial_b \partial^b \bar{h}_{ac} + O(\varepsilon^2)$$

$$= -\frac{\varepsilon}{2} \nabla^2 \bar{h}_{ac} + O(\varepsilon^2)$$

9.3 - Newtonian limit: Einstein equation

Assume pressureless matter at rest in the approx. inertial frame

$$T_{ab} = \rho u_a u_b \,, \qquad u_a = (1,0,0,0)$$

So $G_{ab} = \lambda T_{ab}$ is

$$-\frac{1}{2} \varepsilon \underline{\nabla}^2 \bar{h}_{ab} = \lambda T_{ab}$$

$$\begin{cases} -\frac{1}{2} \varepsilon \underline{\nabla}^2 \bar{h}_{00} = \lambda \rho \\ \underline{\nabla}^2 \bar{h}_{ab} = 0 \qquad (ab) \neq (00) \end{cases}$$

2nd equation: if $\bar{h}_{ab} \to 0$ at infinity, need

$$\bar{h}_{ab} = 0 \qquad (ab) \neq (00)$$

$$\Rightarrow \bar{h} = \eta^{ab} \bar{h}_{ab} = -\bar{h}_{00}$$

$$\Rightarrow h_{00} = \bar{h}_{00} - \frac{1}{2} \eta_{00} \bar{h}$$

$$= \frac{1}{2} \bar{h}_{00}$$

So 1st eq^{n} is: $\underline{\nabla}^2 h_{00} = -\frac{\lambda}{\varepsilon} \rho$

Compare : $\underline{\nabla}^2 \phi = 4\pi \rho G$

- how is ϕ related to $\bar{h}_{00}$?

9.4 - Newtonian limit : geodesic equation.

$$\ddot{x}^a + \Gamma^a{}_{bc}\, \dot{x}^b \dot{x}^c = 0$$

$$\dot{x}^a = \frac{dx^a}{d\tau}$$ "proper time reduces to coordinate time in Newtonian limit"

Assume non-relativistic motion :

$$\dot{x}^a = (1, 0, 0, 0) + \mathcal{O}(\varepsilon)$$

$\tau \simeq$ coordinate time x^0 of approx. inertial frame.

$$\Rightarrow \quad \frac{d^2 x^a}{dt^2} = -\Gamma^a{}_{00}$$

$$\Rightarrow \quad \frac{d^2 x^i}{dt^2} = -\Gamma^i{}_{00} = +\frac{\varepsilon}{2} \partial_i h_{00}$$

Compare : $\ddot{x}^i = -\partial_i \phi \quad \Rightarrow \quad \phi = -\frac{\varepsilon}{2} h_{00} = -\frac{\varepsilon}{4} \bar{h}_{00}$

9.5- Value of λ

Plug back into Einstein equation

$$\Rightarrow \quad \underline{\nabla}^2 \phi = \frac{\lambda}{2} \rho = 4\pi G \rho$$

$$\Rightarrow \quad \lambda = 8\pi G$$

$$\Rightarrow \quad R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G \, T_{ab}$$