Proof: for a timelike geodesic, $\sqrt{-g(\dot{x},\dot{x})} > 0$ along curve so equivalent to minimising

$$S = \int_{\lambda_1}^{\lambda_2} \mathcal{L} \, , \qquad \mathcal{L} = g_{ab}(x)\, \dot{x}^a \dot{x}^b$$

NB: $\Delta\tau$ functional invariant under any reparametrisation of λ.

S invariant under $\lambda \mapsto a\lambda + b$ "affine"

S minimised for $\mathcal{L}$ satisfying Euler-Lagrange equations

$$\text{EL}: \quad \frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{x}^a} - \frac{\partial \mathcal{L}}{\partial x^a} = 0$$

$$\frac{\partial \mathcal{L}}{\partial x^a} = \partial_a g_{bc}\, \dot{x}^b \dot{x}^c$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^a} = 2 g_{ab}\, \dot{x}^b$$

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^a}\right) = 2\left(\frac{d}{d\lambda} g_{ab}\right) \dot{x}^b + 2 g_{ab}\, \ddot{x}^b$$

$$= 2 \dot{x}^c \partial_c g_{ab}\, \dot{x}^b + 2 g_{ab}\, \ddot{x}^b$$

$$\Rightarrow 0 = g_{ab}\ddot{x}^b + \partial_c g_{ab}\dot{x}^c\dot{x}^b - \frac{1}{2}\partial_a g_{bc}\dot{x}^b\dot{x}^c$$

$$= g_{ab}\ddot{x}^b + \frac{1}{2}\left(\partial_c g_{ab} + \partial_b g_{ac} - \partial_a g_{bc}\right)\dot{x}^b\dot{x}^c$$

$$\Rightarrow 0 = \ddot{x}^a + \frac{1}{2}g^{ad}\left(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc}\right)\dot{x}^b\dot{x}^c$$

$$\Rightarrow \quad \ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0$$

Comments : 1) Levi-Civita connection appears naturally

2) In practice, comparing geodesic equation with $\delta S = 0$ equations is easiest way to compute $\Gamma^a{}_{bc}$.

Example : Flat $\mathbb{R}^2$ in polar coordinates (PS2)

Example : S^2 with round metric

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

$$\mathcal{L} = \dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \partial_\theta \mathcal{L} = 2\sin\theta\cos\theta\,\dot{\phi}^2 \qquad \partial_\phi \mathcal{L} = 0$$

$$\partial_{\dot{\theta}}\mathcal{L} = 2\dot{\theta} \qquad \partial_{\dot{\phi}}\mathcal{L} = 2\sin^2\theta\,\dot{\phi}^2$$

θ : $\ddot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0$

ϕ : $\frac{d}{d\lambda}\left(\sin^2\theta\,\dot{\phi}\right) = 0$ "conservation of ang. momentum"

$\Rightarrow \ddot{\phi} + 2\frac{\cos\theta}{\sin\theta}\dot{\phi}\dot{\theta} = 0$

Compare with $\ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0$

$\Rightarrow \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$

$\Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \frac{\cos\theta}{\sin\theta}$

Comment : L did not depend on ϕ so there was a conserved quantity along the geodesic

What do the geodesics look like?

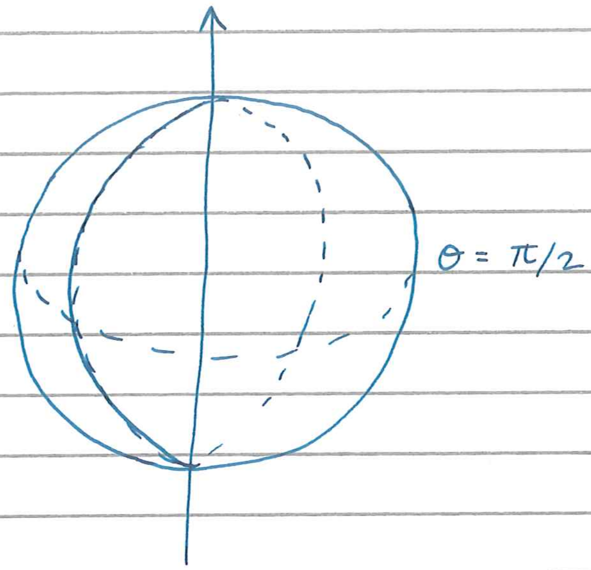
- Choose $\dot{\phi}(0) = 0$ (no ϕ motion)

$\Rightarrow \ddot{\phi} = 0$ from ϕ equation

$\Rightarrow \phi(\lambda) =$ constant (as $\dot{\phi}(0) = \dot{\phi}(\lambda) = 0$)

$\Rightarrow \ddot{\theta} = 0$ from θ equation

So $\dot{\theta}$ = constant "Great circles"



7.1 - Review

Spacetime : 1) Smooth manifold M

2) Metric tensor $g_{ab}(x)$

Unique covariant derivative D_a : 1) $D_a g_{bc} = 0$

2) $\Gamma^c{}_{ab} = \Gamma^c{}_{ba}$

Geodesics : 1) Curve $x^a(\lambda)$ with $T^a = \dot{x}^a(\lambda)$

$$T^a D_a T^b = 0$$

2) Minimise proper time $\Delta\tau$

7.2 - Local inertial frames

Can always find coordinates around a point P such that

1) $x^a(p) = 0$

2) $\Gamma^a{}_{bc}(x) = 0 + O(x)$

3) $g_{ab}(x) = \eta_{ab} + O(x^2)$

Example : S^2 with round metric

$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

let $\tilde{\theta} = \theta - \pi/2$, $\tilde{\phi} = \phi - c$

So $(\tilde{\theta}, \tilde{\phi}) = (0,0)$ at $\phi = c$ on equator

$$\Rightarrow ds^2 = d\tilde{\theta}^2 + \cos^2\tilde{\theta} \, d\tilde{\phi}^2$$

$$= d\tilde{\theta}^2 + \left(1 - \frac{\tilde{\theta}^2}{2}\right)^2 d\tilde{\phi}^2$$

$$= d\tilde{\theta}^2 + d\tilde{\phi}^2 + O(\tilde{\theta}^2)$$

Metric looks flat to $O(x^2)$

In local inertial frame, geodesic equation is

$$\frac{d^2 x^a}{d\lambda^2} = 0 + O(x^3)$$

so free particles move in straight lines (as in SR).

These local coordinates are those used by a freely falling observer.

In a local frame, $\nabla_a = \partial_a$.

Suggests that we can promote equations from SR to GR using

$$\partial_a \to D_a \qquad \text{"minimal coupling"}$$

which then reduce to SR laws in a local inertial frame.

<u>Example</u>: $D_a F^{ab} = 4\pi J^b$

<u>Electromagnetism</u>

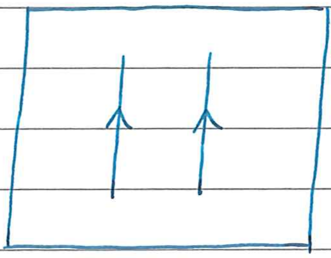
$D_{[a} F_{bc]} = 0$

$D_a T^{ab} = 0$ <u>Stress tensor</u>

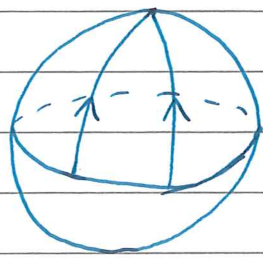
<u>7.3 - Curvature</u>

<u>Deviations</u> from special relativity at $O(x^2)$ in g_{ab}

<u>Characterised</u> by <u>focussing</u> (or parting) of geodesics



$\mathbb{R}^2$



S^2