2. Review of Special Relativity

Model spacetime on $\mathbb{R}^4$

- Flat "Minkowski" space M_4.
- Points in M_4 are "events"
- Label points with $x^\mu = (t, x^i) = (x^0, x^i)$ (but no invariant meaning)

Conventions: use "summation convention"

- Inertial coordinates $(\mu, \nu, \ldots)$
- General coordinates $(a, b, \ldots)$
- Euclidean coordinates $(i, j, \ldots)$

M_4 has natural measure of distance

$$ds^2 = -(dx^0)^2 + \sum_i (dx^i)^2$$

$$= \eta_{\mu\nu}\, dx^\mu dx^\nu$$

$$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$$

Coordinates in which this holds are inertial frames

Lorentz transformations

Linear transformations of x^μ that leave ds^2 invariant

$$x^\mu \mapsto x'^\mu = L^\mu{}_\nu x^\nu$$

$$ds^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu \equiv \eta_{\mu\nu}\, dx'^\mu\, dx'^\nu$$

$$\Rightarrow \quad \eta_{\mu\nu} L^\mu{}_\sigma L^\nu{}_\rho = \eta_{\sigma\rho}$$

In matrix notation

$$x' = L x \,, \qquad L^T \eta L = \eta$$

These transformations relate "inertial frames" in which speed of light is the same.
Their defining property implies

- $\det L = \pm 1$

- $L \in O(1,3)$

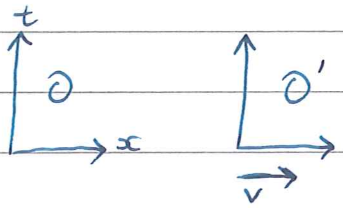
Minkowski version of $R^T \mathbb{1} R = \mathbb{1}$ for $O(4)$

Elements can be broken into "boosts" and "rotations"

Boosts :- "rotations" which mix space and time

- relate frames moving relative to each other

e.g.



$$x'^{\mu} = L^{\mu}{}_{\nu}\, x^{\nu}$$

$$L^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma v & & \\ -\gamma v & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\gamma = (1 - v^2)^{-1/2}$$

Rotations : Only in $\mathbb{R}^3 \subset \mathbb{R}^{1,3}$, $\quad x'^{i} = R^{i}{}_{j}\, x^{j}$

$$L^{\mu}{}_{\nu} = \begin{pmatrix} 1 & \\ & R^{i}{}_{j} \end{pmatrix}, \qquad R \in SO(3)$$

Discrete symmetries : "Parity" $\quad x^{i} \mapsto -x^{i}$

"Time reversal" $\quad x^{0} \to -x^{0}$

Generally, can also shift the coordinates so that

$$x'^{\mu} = L^{\mu}{}_{\nu}\, x^{\nu} + c^{\mu}$$

constant

Together, these give Poincaré transformations.

Causal structure

Metric η defines Lorentz invariant distance between events P and Q, with coordinates x_P^μ and x_Q^μ

$$(\Delta x)^2 = \eta_{\mu\nu} (x^\mu_P - x^\mu_Q)(x^\nu_P - x^\nu_Q)$$

$$= \eta(x_P - x_Q, x_P - x_Q)$$

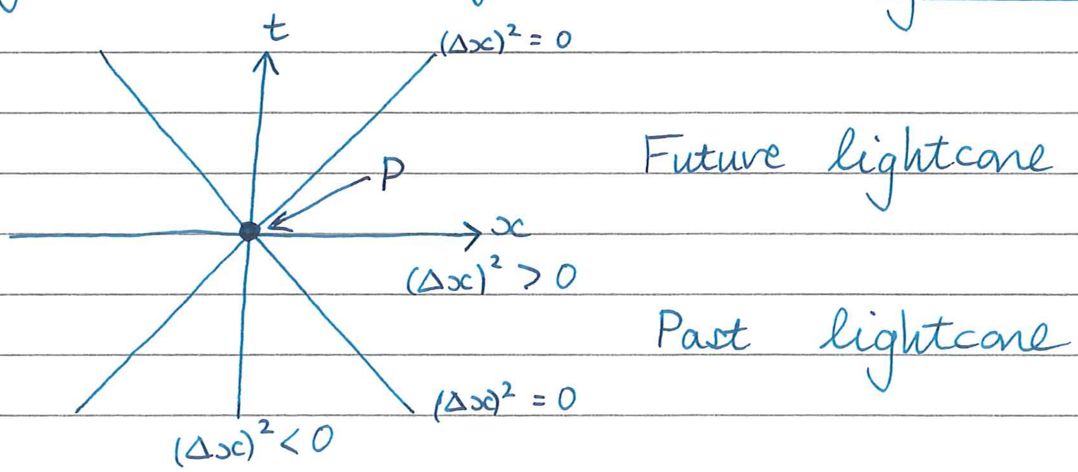
Depending on the sign of $(\Delta x)^2$, the events P and Q are called

$(\Delta x)^2 < 0$ "timelike separated"

$(\Delta x)^2 = 0$ "lightlike separated"

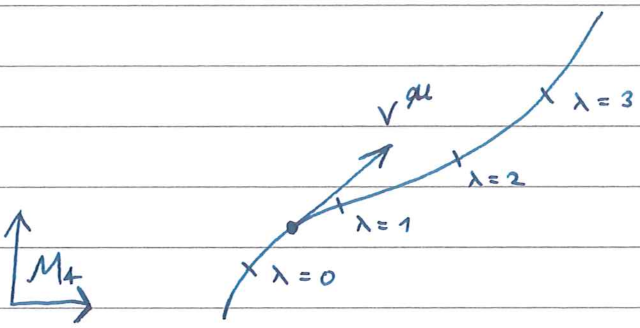
$(\Delta x)^2 > 0$ "spacelike separated"

The set of events that are lightlike separated from P define the "lightcone" at P



Curves and tangent vectors

Curve given by a map $\lambda \mapsto x^\mu(\lambda)$ $\quad(\mathbb{R} \to M_4)$



Tangent vector to curve at $x^\mu(\lambda_0)$ is

$$v^\mu = \frac{d}{d\lambda} x^\mu(\lambda)\Big|_{\lambda=\lambda_0}$$

The tangent vector v^μ is called

$\eta(v,v) < 0$ "timelike"

$\eta(v,v) = 0$ "null"

$\eta(v,v) > 0$ "spacelike"

The sign of $\eta(v,v)$ depends on the image of the curve, not its parametrisation.

A curve whose tangent vector is everywhere timelike is a "timelike curve", etc.

- Massive particles follow timelike curves
- Massless particles follow null curves.

Proper time

A natural parametrisation for a timelike curve is given by the Lorentz invariant "proper time" τ along the curve.

$$x^\mu = x^\mu(\tau)$$

$$d\tau = \sqrt{-ds^2}$$

$$= \sqrt{-\eta_{\mu\nu}\, dx^\mu\, dx^\nu}$$

$$= \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}\, d\tau$$

$$\Rightarrow \quad -\eta_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu = 1 \qquad \left(\dot{x}^\mu := \frac{dx^\mu}{d\tau} \right)$$

Similarly, spacelike curves parametrised by "proper distance" ds.

As τ is Lorentz invariant

$$\dot{x}'^\mu(\tau) = \frac{d}{d\tau} x'^\mu(\tau) = \frac{\partial x'^\mu}{\partial x^\nu} \frac{dx^\nu}{d\tau} = L^\mu{}_\nu\, \dot{x}^\nu(\tau)$$

Lorentz vectors (4-vectors)

Objects with components v^μ that transform under Lorentz transformations as

$$v^\mu \mapsto v'^\mu = L^\mu{}_\nu v^\nu$$

The metric $\eta_{\mu\nu}$ gives an indefinite scalar product on vectors.

The scalar product $\eta_{\mu\nu} v^\mu w^\nu$ is a Lorentz scalar, so is invariant under Lorentz transformations

$$\eta_{\mu\nu} v'^\mu w'^\nu = \eta_{\mu\nu} L^\mu{}_\rho v^\rho L^\nu{}_\sigma w^\sigma \quad L^T \eta L = \eta$$
$$= \eta_{\mu\nu} v^\mu w^\nu$$

A vector is timelike, null, ... depending on sign of $\eta(v,v)$.

Other Lorentz tensors

Scalars are invariant, e.g. $\eta(v,w)$

Co-vectors (or 1-forms) transform under dual representation.

$$\Lambda = (L^T)^{-1} = \eta L \eta^{-1}$$

$$u'_\mu = \Lambda_\mu{}^\nu u_\nu \, , \qquad \Lambda_\mu{}^\nu = \eta_{\mu\rho} L^\rho{}_\sigma \eta^{\sigma\nu}$$
$$= u_\nu (L^{-1})^\nu{}_\mu$$

Vectors are elements of $T_p M_4$. (at a point p)

Covectors are elements of $T_p^* M_4$

Covectors define a map $T_p M_4 \to \mathbb{R}$

$$u : v \in T_p M_4 \mapsto u(v) = u_\mu v^\mu \in \mathbb{R} \, .$$

The metric $\eta_{\mu\nu}$ defines an isomorphism

$$T_p M_4 \cong T_p^* M_4$$

$$v^\mu = \eta^{\mu\nu} v_\nu \, , \qquad v_\mu = \eta_{\mu\nu} v^\nu$$

(p,q) tensors transform like a product of vectors and covectors

$$T^{\mu_1 .. \mu_p}{}_{\nu_1 .. \nu_q} \mapsto T'^{\mu_1 .. \mu_p}{}_{\nu_1 .. \nu_q}$$
$$= L^{\mu_1}{}_{\rho_1} \dots L^{\mu_p}{}_{\rho_p} (L^{-1})^{\sigma_1}{}_{\nu_1} \dots (L^{-1})^{\sigma_q}{}_{\nu_q} T^{\rho_1 .. \rho_p}{}_{\sigma_1 .. \sigma_q}$$

Products e.g. $v^\mu w^\nu u_\rho$ are $(2,1)$ tensor

- Linear combinations of (p,q) tensors are (p,q).
- Products and contractions of tensors are again tensors.
- Type of tensors can be read off from indices.

Tensor fields

Assignment of a tensor to each point of M_4

$$T : x \mapsto T^{\mu_1 .. \mu_p}{}_{\nu_1 .. \nu_q}(x)$$

e.g. given $V^\mu(x)$, $\eta_{\mu\nu} V^\mu(x) V^\nu(x) = \eta(V,V)(x)$ is a scalar function.

e.g. given scalar function $f(x)$

$$u_\mu(x) := \frac{\partial}{\partial x^\mu} f(x) \equiv \partial_\mu f$$

is a covector / 1-form.

e.g. Wave operator $\Box = \eta^{\mu\nu} \partial_\mu \partial_\nu$ maps (p,q) to (p,q)

$(\nabla^2 = \Box)$

e.g. $\eta_{\mu\nu}$ is a $(0,2)$ tensor that does not depend on x or change with Lorentz transformations

i.e. $\partial_\mu \eta_{\nu\rho} = 0$, $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$

e.g. Identity tensor / Kronecker delta

$\delta^\mu{}_\nu = \mathrm{diag}(1,1,1,1)$ in any inertial frame.

$\delta^\mu{}_\nu T^{\nu\cdots}{}_{\cdots} = T^{\mu\cdots}{}_{\cdots}$

$\eta^{\mu\rho}\eta_{\nu\rho} = \delta^\mu{}_\nu$

e.g. $T_{[\mu\nu]} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu})$

$T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$

<u>Worldlines of massive particles</u>

Timelike curve parametrised by <u>proper time τ</u>

$u^\mu = \dot{x}^\mu(\tau)$ "4-velocity"

$u^\mu u_\mu = \eta_{\mu\nu} u^\mu u^\nu = -1$

e.g. $u^\mu = (1, \vec{0})$ at rest

$u^\mu = \gamma(v)(1, \vec{v})$ massive

$u^\mu = (1, \vec{v})$, $|\vec{v}|^2 = 1$ massless

<u>4-acceleration</u>: $a^\mu = \frac{d}{d\tau} u^\mu$

Massive free particles obey $a^\mu = 0$