Numerical Linear Algebra. QS 1 *(MT 2019)*

Optional question: Qn 4

1. Show that $H(w)^2 = I$ when H is a Householder matrix.

2. Show that $\|x\|_\infty = \max_i |x_i|$ satisfies the axioms for a vector norm.

3. Show that if $\|x\|$ is a vector norm then $\sup_x \frac{\|A\,x\|}{\|x\|}$ satisfies the axioms for a matrix norm. Further show that

$$\|A\,B\| \leq \|A\|\,\|B\|.$$

4. From the definition of the vector 1–norm show that

$$\|A\|_1 = \max_j \sum_i |a_{ij}|.$$

5. By considering the individual columns a_j of A and b_j of $B = Q\,A$, show that

$$\|Q\,A\|_{\mathrm{F}} = \|A\|_{\mathrm{F}}$$

if Q is an orthogonal matrix.

6. By using the definition of the vector 2–norm and the SVD show that

$$\|A\|_2 = \sigma_1$$

where σ_1 is the largest singular value.

7. (a) For $A \in \mathbb{R}^{m\times n}$ show that the singular values of A are the square roots of the eigenvalues of $A^{\mathrm{T}}\,A$ if $m \geq n$ or of $A\,A^{\mathrm{T}}$ is $m \leq n$. (You might want to consider what A and A^{T} do to the singular vectors.)
(b) Check the above using matlab: e.g. set

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$lam = eig(A' * A)$ and $sing = svd(A)$.

8. If $A \in \mathbb{R}^{n\times n}$ is nonsingular, what is the SVD of A^{-1} in terms of that of A?