$$e_j = (M^{-1}N)^{\mu}(A^{-1} - P\overline{A}^{-1}R)\ \ A\ (M^{-1}N)^{\nu}e_{j-1}$$

Convergence depends on

- Smoothing (as above)
- Approximation: R and P must sufficiently accurately reproduce smooth vectors and $\overline{A}$ sufficiently accurately represent A

In mathematical terms: we use the regular Euclidean norm $\|\cdot\|$ and $\|\cdot\|_A$ defined by $\|x\|_A^2 = x^T A x$ which is a norm when A is symmetric and positive definite, and establish

- the Smoothing Property: for all y

$$\|A\,(M^{-1}N)^\nu y\| \leq \eta(\nu)\ \|y\|_A$$

  with $\eta(\nu) \to 0$ as $\nu \to \infty$ being independent of n ($n =$dimension of A)

- the Approximation Property: for all y, there exists C independent of n with

$$\|(A^{-1} - P\overline{A}^{-1}R)y\|_A \leq C\|y\|.$$

Immediately we have

Theorem: if the Smoothing and Approximation properties hold then 2-grid iteration with no post-smoothing converges at a rate independent of n.

Proof:

$$\begin{aligned}
\|e_j\|_A &= \|(A^{-1} - P\overline{A}^{-1}R)\ A\ (M^{-1}N)^\nu e_{j-1}\|_A \\
&\le C\|A\ (M^{-1}N)^\nu e_{j-1}\| \qquad \text{(Approx. Property)} \\
&\le \underbrace{C\eta(\nu)}_{\to 0 \text{ as } \nu\to\infty} \quad \|e_{j-1}\|_A \qquad \text{(Smoothing Property)}
\end{aligned}$$

so $\exists$ a number of smoothing steps ν independent of n with

$$\|e_j\|_A \le \gamma\|e_{j-1}\|_A$$

with $\gamma < 1$. $\square$

Approximation Property: rough sketch (depends on finite difference error)

$$\begin{aligned} A^{-1}b &\leftrightarrow \text{mesh solution on mesh } h \leftrightarrow u_h \\ P\overline{A}^{-1}Rb &\leftrightarrow \text{mesh solution on mesh } 2h \leftrightarrow u_{2h} \end{aligned}$$

so $\|(A^{-1} - P\overline{A}^{-1}R)b\|_A \sim |u_h - u_{2h}| \sim \|b\|$ any b.

Smoothing Property: we prove only for relaxed Jacobi :

$$M^{-1}N = (1-\theta)I - \theta D^{-1}(L+U) = I - \theta D^{-1}A = I - \frac{\theta}{4}A$$

as $D = 4I$ for 5 point formula.

Theorem: if the eigenvalues of $M^{-1}N$ lie in $[-\sigma, 1]$ with $0 \leq \sigma < 1$ being independent of n, then the smoothing property holds.

(Recall: $\theta = \frac{1}{2} \Rightarrow$ eigenvalues of $M^{-1}N \in (0, 1)$.)

Proof: let $\{z_1, \ldots, z_n\}$ be the orthonormal eigenvector basis of $I - (\theta/4)A$ and $y = \sum c_i z_i$, $(I - (\theta/4)A)z_i = \lambda_i z_i$. Then $Az_i = (4/\theta)(1 - \lambda_i)z_i$ so

$$A(M^{-1}N)^\nu y \;=\; (4/\theta) \sum c_i \lambda_i^\nu (1 - \lambda_i) z_i,$$

and $$\|A(M^{-1}N)^\nu y\|^2 \;=\; (16/\theta^2) \sum c_i^2 \lambda_i^{2\nu} (1 - \lambda_i)^2$$

as $z_i^T z_j = \delta_{i,j}$. Now $\lambda_i \in [-\sigma, 1] \Rightarrow \lambda_i^{2\nu}(1 - \lambda_i)$ is maximal either at the stationary point $\lambda_i = 2\nu/(2\nu + 1)$ or when $\lambda_i = -\sigma$ so that

$$\max_{\lambda_i \in [-\sigma, 1]} \lambda_i^{2\nu}(1 - \lambda_i) \le \max \left\{ \frac{1}{2\nu} \, \frac{1}{e}, \sigma^{2\nu}(1 + \sigma) \right\}$$

since $$\left(\frac{2\nu}{2\nu + 1} \right)^{2\nu} \frac{1}{2\nu + 1} = \frac{1}{2\nu} \, \frac{1}{(1 + 1/2\nu)^{2\nu + 1}} \le \frac{1}{2\nu} \, \frac{1}{e}$$

and $(1 + 1/2\nu)^{2\nu+1} \searrow e (= 2.718\ldots)$ as $\nu \to \infty$.

Thus

$$\|A(M^{-1}N)^{\nu}y\|^2$$

$$\begin{aligned} &\leq \max\left\{\frac{4}{e\theta}\frac{1}{2\nu}, \frac{4}{\theta}\sigma^{2\nu}(1+\sigma)\right\}\sum c_i^2\frac{4}{\theta}(1-\lambda_i) \\ &= \underbrace{\max\left\{\frac{4}{e\theta}\frac{1}{2\nu}, \frac{4}{\theta}\sigma^{2\nu}(1+\sigma)\right\}}_{\eta(\nu)\to 0 \text{ as } \nu\to\infty}\|y\|_A^2 \end{aligned}$$

Notes:

- Can be extended to Multigrid by replacing the coarse grid solve $\overline{A}\,\overline{e}^s = \overline{r}^s$ recursively by a 2-grid iteration: just apply Gauss Elimination when very small dimensional coarse space. n-independent convergence is preserved.
- Other smoothers,prolongation and restriction operators and more general problems can be analysed in a similar way.
- Work per iteration depends linearly on problem size (n^2). Number of iterations for convergence independent of $n \Rightarrow$ optimal solver (ie. $O(N)$ work to solve an $N \times N$ linear system.
  (cf. $O(N^3)$ for Gauss Elimination).