Numerical Linear Algebra. QS 4 *(MT 2019)*

Optional questions: Qns 5, 8

1. (a) For the 5–point finite difference approximation of the Laplacian on $[0,1]\times[0,1]$ with Dirichlet Boundary Conditions on a regular mesh of size h with $(n+1)\,h = 1$, verify that for each $r,\, s \in \{1,\cdots,n\}$, the vector $U^{r\,s}$ with components $u^{rs}_{jk} = \sin r\,j\,\pi\,h\,\sin s\,k\,\pi\,h$ is an eigenvector of the Jacobi iteration matrix with corresponding eigenvalue $\lambda^{rs} = \frac{1}{2}\,(\cos r\,\pi\,h + \cos s\,\pi\,h)$.

(b) Show that μ, v are an eigenvalue and eigenvector of the Gauss–Siedel iteration matrix for $A = D + L + U$, (D is a diagonal matrix, L is a strictly lower triangular matrix, U is a strictly upper triangular matrix), if and only if $(\mu\,D + \mu\,L + U)\,v = 0$. For the 5–point problem above show that v^{rs} with components $v^{rs}_{jk} = \mu^{\frac{j+k}{2}}\,\sin r\,j\,\pi\,h\,\sin s\,k\,\pi\,h$ is an eigenvector of the G–S iteration matrix. What is the corresponding eigenvalue? Which of Jacobi or G–S iteration will converge more rapidly for this problem? Is G–S suitable for use as a smoother in multigrid?

2. Consider using multigrid to solve the 1–dimensional boundary value problem,

$$-u'' = f, \quad x \in [0,1], \quad u(0) = 0 = u(1)$$

approximately by the finite difference

$$-u_{j-1} + 2\,u_j - u_{j+1} = h^2\,f, \quad j = 1, 2, \ldots, n$$

on a regular grid $jh, j = 0, 1, \ldots, n+1$ with $(n+1)\,h = 1$ and n odd.

If $u^e = (u_2, u_4, \cdots, u_{n-1})^T$ is a vector of the coarse grid unknowns, show that prolongation to fine grid values $(u_1, u_2, \cdots, u_n)^T$ by linear interpolation between adjacent points can be represented by $P\,u^e$ where

$$P = \begin{bmatrix} \frac{1}{2} & & & & \\ 1 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ & 1 & & & \\ & \frac{1}{2} & \frac{1}{2} & & \\ & & 1 & \cdot & \\ & & & \cdot & \\ & & & & \cdot \\ & & & \frac{1}{2} & \frac{1}{2} \\ & & & & 1 \\ & & & & \frac{1}{2} \end{bmatrix} \in \mathbb{R}^{n \times \frac{n-1}{2}}$$

What restriction does $R = \frac{1}{2}\,P^T$ represent?

3. Consider a two-grid iteration for $A\mathbf{u} = \mathbf{f}$ with no post-smoothing. Here P is the prolongation, R the restriction and the smoothing iteration is based on the splitting $A = M - N$. Denote by $\mathbf{u}_s^{(i)}$ the result of k pre-smoothing steps on the i^{th} multigrid iterate $\mathbf{u}^{(i)}$. Show that the next two-grid iterate is

$$\mathbf{u}^{(i+1)} = \mathbf{u}_s^{(i)} + P\overline{A}^{-1}R(\mathbf{f} - A\mathbf{u}_s^{(i)}).$$

By further showing that

$$\mathbf{f} - A\mathbf{u}_s^{(i)} = A(I - M^{-1}A)^k(\mathbf{u} - \mathbf{u}^{(i)}),$$

demonstrate that

$$\mathbf{u} - \mathbf{u}^{(i+1)} = (A^{-1} - P\overline{A}^{-1}R)A(I - M^{-1}A)^k(\mathbf{u} - \mathbf{u}^{(i)}).$$

4. Similarly to problem 3, show that if m post-smoothing iterations are also employed then

$$\mathbf{u} - \mathbf{u}^{(i+1)} = (I - M^{-1}A)^m(A^{-1} - P\overline{A}^{-1}R)A(I - M^{-1}A)^k(\mathbf{u} - \mathbf{u}^{(i)}).$$

5. Show that in two-grid iteration the residual $\mathbf{r}^{(i)} = A\mathbf{e}^{(i)}$ satisfies

$$\mathbf{r}^{(i+1)} = (I - AM^{-1})^m A(A^{-1} - P\overline{A}^{-1}R)(I - AM^{-1})^k\mathbf{r}^{(i)}.$$

6. By writing the two-grid iteration matrix

$$(I - M^{-1}A)^m(A^{-1} - \alpha P\overline{A}^{-1}P^T)A(I - M^{-1}A)^k$$

for a symmetric matrix A, with restriction equal to αP^T for some real positive α, in the form $I - M_{MG}^{-1}A$, show that M_{MG} is symmetric when $k = m$ and M is symmetric. You should assume that $\overline{A}$ is symmetric. It follows that two-grid iteration corresponds to a symmetric splitting $A = M_{MG} - N$ under these conditions.

7. Similarly to problem 6, show that provided $k = m$, then employing M^T in the post-smoother and M in the pre-smoother results in a symmetric two-grid splitting matrix M_{MG} when M is non-symmetric. This is of relevance when for example Gauss-Seidel smoothing is used since M^T then corresponds to reversing the ordering of the variables.

8. In this question the restriction is $R = \alpha P^T$ where P is the prolongation and the Galerkin coarse grid operator $\overline{A} = \alpha P^T A P$ is used. Writing

$$G^{pre} = (A^{-1} - \alpha P \overline{A}^{-1} P^T) A (I - M^{-1} A)^k$$

for the two-grid iteration matrix when only pre-smoothing is used and

$$G^{post} = (I - M^{-1} A)^m (A^{-1} - \alpha P \overline{A}^{-1} P^T) A$$

for the two-grid iteration matrix when only post-smoothing is used, show that

$$G^{post} G^{pre} = (I - M^{-1} A)^m (A^{-1} - \alpha P \overline{A}^{-1} P^T) A (I - M^{-1} A)^k$$

is the standard two-grid iteration matrix with m post-smoothing and k pre-smoothing iterations. It follows that if $\|G^{pre}\| < 1$ and $\|G^{post}\| < 1$ then the standard two-grid iteration is convergent in the same norm.

9. Show that for the 5-point matrix, relaxed Jacobi smoothing with $\theta = 4/5$ ensures that all of the eigenvalues with $r > n/2$ *or* $s > n/2$ (which correspond to eigenvectors with high frequency variation in at least one of the coordinate directions) lie in $(-3/5, 3/5]$.