C5.12 [C8.1] (1

Math Phys Biol 2008 q6 answer

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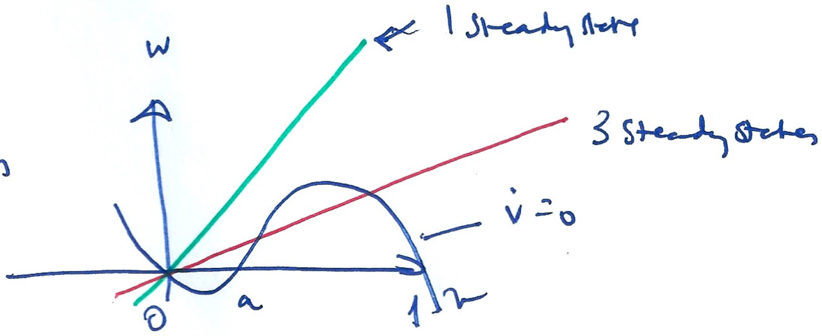
$$\varepsilon \dot{v} = f(v) - w + I \qquad f = v(a-v)(v-1)$$
$$\dot{w} = bv - \gamma w \qquad 0<a<1$$

v is fast because $\varepsilon \ll 1$! [ie. so $\dot{v} \gg 1$ in general...]

(a) $I=0$

(i) Plot the nullclines



Clearly $(0,0)$ is always a steady state

For sufficiently small b/γ there are three steady states.

(ii) b/γ large as in green above

Linear stability of $(0,0)$
$$\begin{pmatrix}\dot{v}\\ \dot{w}\end{pmatrix} \simeq \underbrace{\begin{pmatrix} f'(0)/\varepsilon & -1/\varepsilon \\ b & -\gamma \end{pmatrix}}_{M} \begin{pmatrix} v \\ w \end{pmatrix}$$

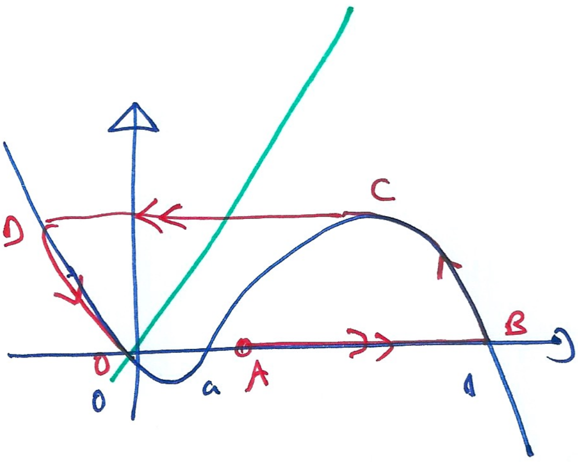
$$\det M = \frac{b}{\varepsilon} - \frac{\gamma f'(0)}{\varepsilon} > 0$$

$$\operatorname{tr} M = \frac{f'(0)}{\varepsilon} - \gamma < 0 \Rightarrow \text{stable (node or spiral)}$$

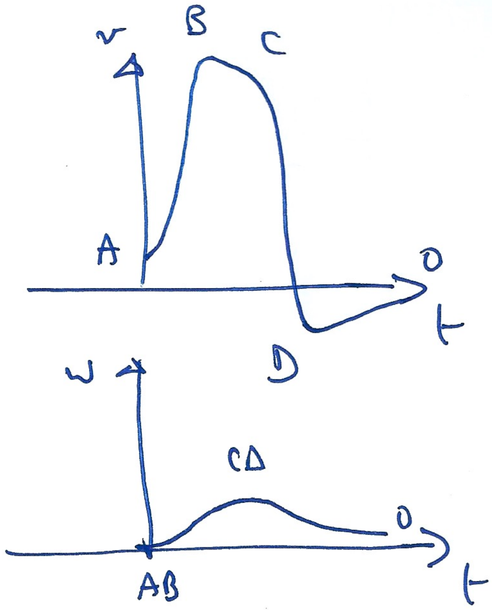
The steady state is excitable : e.g. perturb to $(v>a, 0)$

with $\varepsilon \ll 1$ trajectory is ABCDO as shown

with fast bits AB & CD

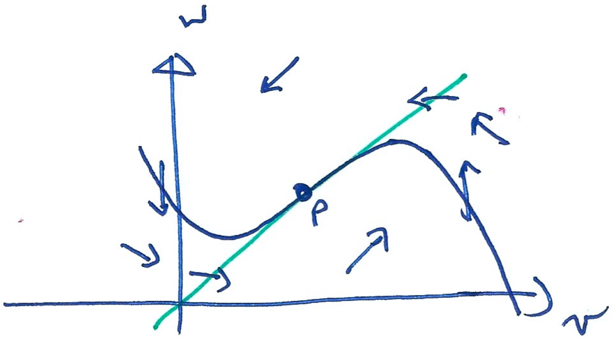


Biologically: indicates generation of nerve impulses under stimulus



(b) $I>0$: v nullcline is $w = f(v) + I$:

e.g.



At large w
$\dot{v}<0, \dot{w}<0$

So trajectories cycle as shown about fixed pt P.

with $v = v^* + V$ etc $\begin{pmatrix}\dot{V}\\ \dot{W}\end{pmatrix} = M\begin{pmatrix}V\\ W\end{pmatrix}$, $M = \begin{pmatrix} f'/\varepsilon & -1/\varepsilon \\ b & -\gamma \end{pmatrix}$, $\operatorname{tr} M = \frac{f'(v^*)}{\varepsilon} - \gamma > 0$

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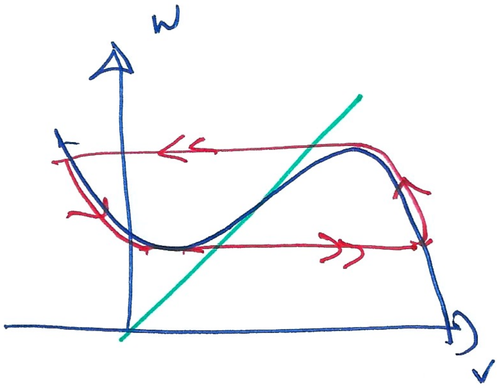
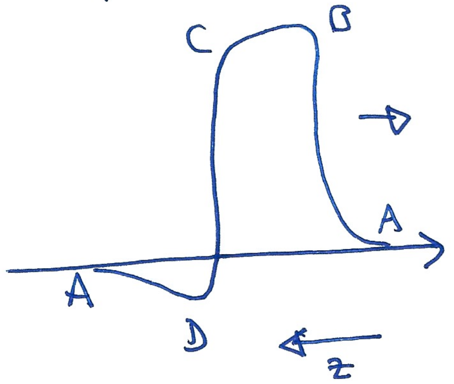
So fixed pt is unstable ⟹ relaxation oscillations as shown



(c) $\varepsilon v_t = f(v) - w + \varepsilon^2 v_{xx}$

$w_t = bv - \gamma w$



(i) $z = ct - x \quad v = v(z), w = w(z)$

$\Rightarrow \varepsilon c v' = f(v) - w + \varepsilon^2 v''$

$c w' = bv - \gamma w$

(ii) AB, $z = \varepsilon X \Rightarrow w' \simeq 0 \Rightarrow w \simeq 0$

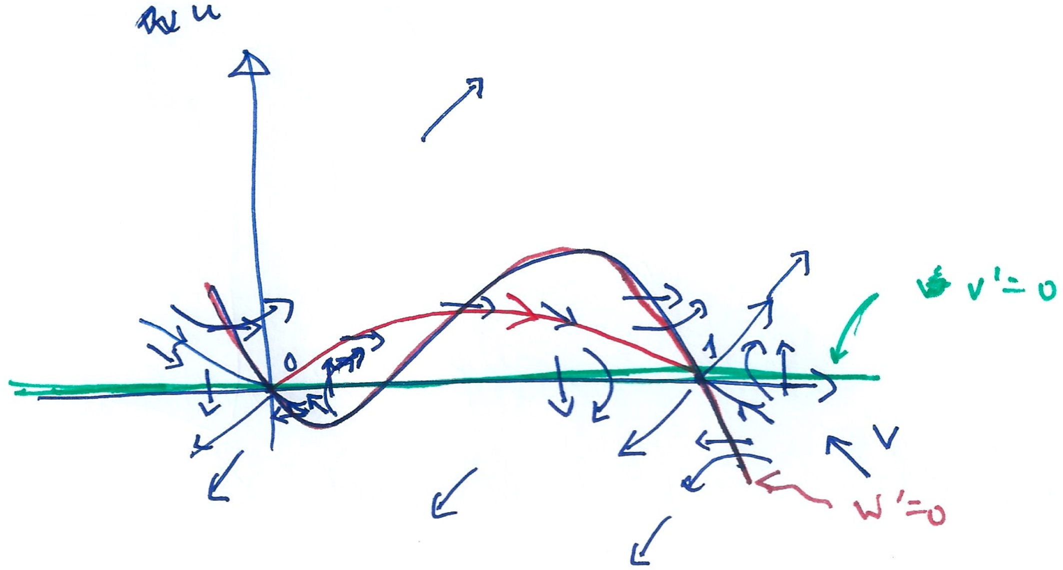
$c v' = f(v) + v'' \qquad v' = \dfrac{dv}{dX}$

A to B takes v from $v = 0$ to $v = 1$ (where $f(v) = 0$)

Phase plane

$$v' = u$$

$$u' = cu - f(v) \qquad c > 0$$



$u \to \infty \quad v', u' > 0$

It is clear that $(0,0)$ & $(1,0)$ are saddles

(& $(a,0)$ is a node/spiral)

Only way to get from $(0,0)$ to $(1,0)$ is out along

unstable separatrix in $v > 0$ from $(0,0)$ & ~~back~~ to $(1,0)$ along

stable separatrix in $v < 1$. As shown in red above

(+ other trajectories in blue)

To connect these we need to select an isolated value

of c. (in fact there is only one)

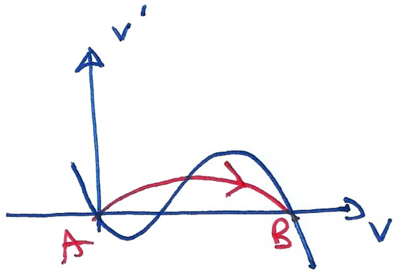
(iii) In slow region BC, DA $\varepsilon \to 0 \Rightarrow w \approx f(v)$

(iv) The full trajectory is then as shown ~~trajectories on the~~

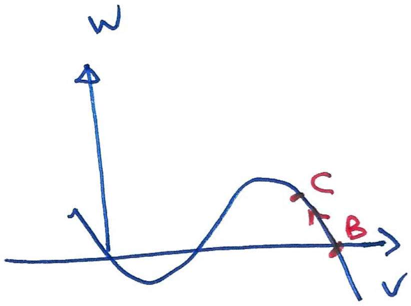
(in sections of

(ie $v, v'=u, w$) space

AB fast

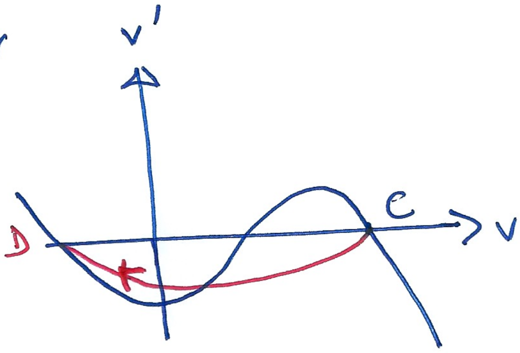


BC slow



go along $w = f(v)$ till ~~...~~ C
$v = v_C$
(to be determined)

CD fast



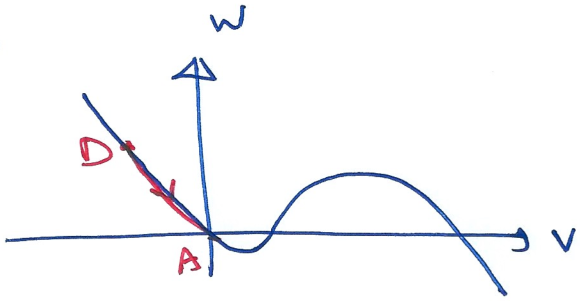
Same phase portrait as AB

but select v_C in order

to connect

separatrices as shown

DA slow



or..