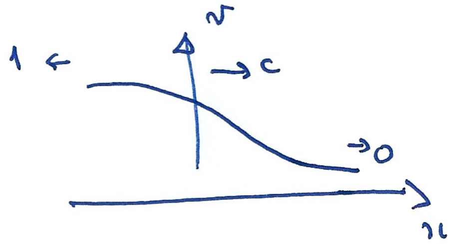
C5.12 (C8.16) Math Physiol 2012 q2

2/ $v_t = \nabla^2 v + v(1-v)f(v)$ $\qquad f>0$



(a) plane waves $\nabla^2 v = v_{xx}$

$v = v(q) \qquad q = x - ct$

$-cv' = v'' + v(1-v)f(v)$

(b) 2-D

$\underline{x} = \underline{X}(\underline{a}, t) \qquad \underline{a} = \underline{A}(\underline{x}, t)$

$\nabla v = \underline{e}_i \frac{\partial v}{\partial x_i}$ & $\frac{\partial}{\partial x_i} = \frac{\partial a_j}{\partial x_i}\frac{\partial}{\partial a_j} = \alpha_{ij}\frac{\partial}{\partial a_j}$

So $\nabla v = \underline{e}_i \alpha_{ij} \frac{\partial v}{\partial a_j}$ summation convention assumed

$$\nabla^2 v = \cancel{\frac{\partial}{\partial x_j}\alpha_{ij}} \frac{\partial}{\partial x_k} \alpha_{kj} \frac{\partial v}{\partial a_j}$$

$$= \alpha_{kl}\frac{\partial}{\partial a_l}\left[\alpha_{kj}\frac{\partial v}{\partial a_j}\right]$$

$$= \underbrace{\alpha_{kl}\alpha_{kj}\frac{\partial^2 v}{\partial a_l \partial a_j}}_{\substack{k \to p \\ l \to i}} + \alpha_{kl}\frac{\partial \alpha_{kj}}{\partial a_l}\frac{\partial v}{\partial a_j}$$

as given with

...

We need to show

$$\alpha_{kl} \frac{\partial \alpha_{kj}}{\partial a_l} \frac{\partial v}{\partial a_j} = \frac{\partial \alpha_{pi}}{\partial x_p} \frac{\partial v}{\partial a_i} + \alpha_{ij} \frac{\partial X_i}{\partial t} \frac{\partial v}{\partial a_j}$$

x_i ||

$$= \frac{\partial \alpha_{kj}}{\partial x_k} \frac{\partial v}{\partial a_j} + \alpha_{kj} \frac{\partial X_k}{\partial t} \frac{\partial v}{\partial a_j}$$

$$= \left(\frac{\partial \alpha_{kj}}{\partial x_k} + \alpha_{kj} \frac{\partial X_k}{\partial t} \right) \frac{\partial v}{\partial a_j}$$

& note $\alpha_{kl} = \dfrac{\partial a_l}{\partial x_k}$

we want to show $\alpha_{kl} \dfrac{\partial \alpha_{kj}}{\partial a_l} = \dfrac{\partial \alpha_{kj}}{\partial x_k} + \alpha_{kj} \dfrac{\partial x_k}{\partial t}$

ugh
well I don't get this. I think he is now considering v as $v(\underline{a}, t)$

So that $\cancel{\frac{\partial v(\underline{x}, t)}{\partial t} = \frac{\partial v}{\partial t} + \frac{\partial v}{}}$ should be V really

$$\left(v(\underline{x}, t) = v(\underline{x}(\underline{a}, t), t) \right) \equiv V(\underline{a}, t)$$

so $\left.\dfrac{\partial v}{\partial t}\right|_{\underline{a}} = \left.\dfrac{\partial v}{\partial t}\right|_{\underline{x}} + \dfrac{\partial x_i}{\partial t} \dfrac{\partial v}{\partial x_i}$

& his required eq

note this is particle velocity

$$\left.\frac{\partial v}{\partial t}\right|_{x} = \left.\frac{\partial v}{\partial t}\right|_{\underline{a}} - \frac{\partial x_i}{\partial t} \frac{\partial v}{\partial x_i} = \nabla^2 v + \dots$$

so $\left.\dfrac{\partial v}{\partial t}\right|_{\underline{a}} = \alpha_{kl} \alpha_{kj} \dfrac{\partial^2 v}{\partial a_l \partial a_j} + \underbrace{\alpha_{kl} \dfrac{\partial \alpha_{kj}}{\partial a_l} \dfrac{\partial v}{\partial a_j}}_{(1)} + \underbrace{\dfrac{\partial x_i}{\partial t} \dfrac{\partial v}{\partial x_i}}_{(2)} + v(1-v)f(v)$

As before

② $$\frac{\partial v}{\partial x_i} = \frac{\partial a_j}{\partial x_i}\frac{\partial v}{\partial a_j} = \alpha_{ij}\frac{\partial v}{\partial a_j}$$

and also $\frac{\partial \alpha_{ki}}{\partial x_k} =$ $\cancel{\alpha_{kj}}$ ① $$\frac{\partial \alpha_{ki}}{\partial a_j}\frac{\partial a_j}{\partial x_k} = \alpha_{kj}\frac{\partial \alpha_{ki}}{\partial a_j}$$

via ①

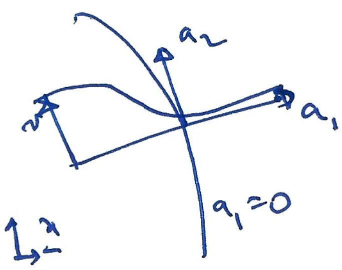
So
$$\left.\frac{\partial v}{\partial t}\right|_{\underline{a}} = \alpha_{kl}\alpha_{kj}\frac{\partial^2 v}{\partial a_l \partial a_j} + \frac{\partial \alpha_{kj}}{\partial x_k}\frac{\partial v}{\partial a_j} + \alpha_{ij}\frac{\partial v}{\partial a_j}\frac{\partial x_i}{\partial t} + v(1-v)f(v)$$

✓ ✓

$k \to p$
$j \to i$

I'm sure there is a prettier way

(c)



$\underline{\alpha} = (\alpha_{11}, \alpha_{21})$, $|\underline{\alpha}| = 1$

$$\underline{\alpha} = \left(\frac{\partial a_1}{\partial x_1}, \frac{\partial a_1}{\partial x_2}\right) = \underline{\nabla} a_1$$

(i) The normal to the wave front is $\underline{\nabla} a_1 = \underline{\alpha}$ ⇒ □.

(ii) We assume v changes rapidly in the front ~~then~~ whose variation along the front is slow.

Specifically we suppose $\frac{\partial}{\partial a_1} \sim 1$ $\frac{\partial}{\partial a_2} \sim \varepsilon$ say

Now note that the curvature is $2\kappa = \underline{\nabla}\cdot\underline{n} = \underline{\nabla}\cdot\underline{\alpha}$

and the front velocity is $\frac{\partial \underline{x}}{\partial t}\cdot\underline{n} = \frac{\partial x_i}{\partial t}\alpha_{i1}$

So approximately we have

$$v_t = \alpha_{p1}\alpha_{p1} v_{a_1 a_1} + \frac{\partial \alpha_{p1}}{\partial x_p} v_{a_1} + \alpha_{i1}\frac{\partial x_i}{\partial t} v_{a_1} + v(1-v)f(v)$$

The front velocity is $v_n = \alpha_{i1}\dfrac{\partial x_i}{\partial t}$

the curvature say $K = 2\kappa = \nabla\cdot\underset{\sim}{\alpha} = \dfrac{\partial \alpha_{i1}}{\partial x_i}$

~~So we have~~ & $\alpha_{p1}\alpha_{p1} = \alpha_{11}^2 + \alpha_{21}^2 = 1$ as $|\underset{\sim}{\alpha}| = 1$

$$\Rightarrow v_t = v_{a_1 a_1} + (K + v_n) v_{a_1} + v(1-v)f(v)$$

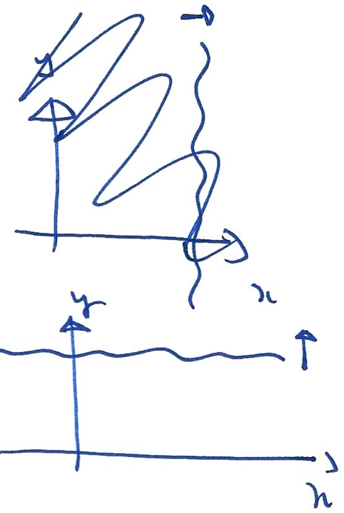
and this has a steady solution (we are in a travelling frame)

iff $K + v_n = c$

Since we know $v'' + cv' + v(1-v)f(v) = 0$ works.

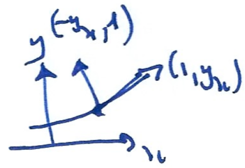
(d) wave front is

$$x = s, \quad y = ct + \delta h(s,t)$$



$$K = \frac{-\delta h_{ss}}{(1+\delta^2 h_s^2)^{3/2}}$$

$$\Rightarrow \quad \frac{-\delta h_{ss}}{(1+\delta^2 h_s^2)^{3/2}} + v_n = c$$



The curve is

$$y = ct + \delta h(x,t)$$

so $y_x = \delta h_x = \delta h_s$

$$\underset{\sim}{n} = \frac{(-\delta h_s, 1)}{(1+\delta^2 h_s^2)^{1/2}}$$

$$\underline{v} = (\dot{x}, \dot{y}) = (0, c + \delta h_t)$$

$$v_n = \frac{c + \delta h_t}{(1+\delta^2 h_s^2)^{1/2}} = F$$

So

$$\frac{-\delta h_{ss}}{(1+\delta^2 h_s^2)^{3/2}} + \frac{c + \delta h_t}{(1+\delta^2 h_s^2)^{1/2}} = c$$

ii

Expanding for small δ

$$-\delta h_{ss} + c + \delta h_t + O(\delta^2) = c$$

$$\Rightarrow h_t = h_{ss}$$

diffusion so stable

[gosh what is this $h \in L^2$?!]

maybe he wants $\frac{d}{dt}\int_{-\infty}^{\infty} h^2\,ds = -\int_{-\infty}^{\infty} h_s^2\,ds < 0$!]