C1

C5.12 Math Physiol 2015 Q2 answer

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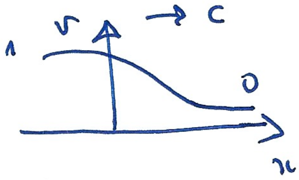
$$\varepsilon v_t = \varepsilon^2 \nabla^2 v + f(v), \quad f = Av(v-a)(1-v)$$



(a) $\underline{x} = \underline{x}(\underline{X}, t)$ Jacobian $\frac{\partial \underline{x}}{\partial \underline{X}} = \underline{J}$, $J_{ij} = \frac{\partial x_i}{\partial X_j}$

inverse is $\underline{X} = \underline{X}(\underline{x}, t)$ Jacobian $(=J^{-1}) = \underline{K}$, $K_{ij} = \frac{\partial X_i}{\partial x_j}$

note $JK = \underline{I}$.

$$\underline{e}_i = \frac{\partial \underline{x}}{\partial X_i}$$

(i) Since X_2 & X_3 define local cartesian coordinates in the interface, if we assume $\underline{X}$ is an orthogonal system, then X_1 is normal to the interface, so at $X_1 = 0$ (on the interface) $\underline{e}_1$ is normal to the interface.

(ii) this is just the material derivative

We have $\left.\frac{\partial v}{\partial t}\right|_{\underline{X}} = \left.\frac{\partial v}{\partial t}\right|_{\underline{x}} + \left.\frac{\partial x_i}{\partial t}\right|_{\underline{X}} \frac{\partial v}{\partial x_i}$ $\quad (v(\underline{x},t) = v[\underline{x}(\underline{X},t),t]$ $(= v(\underline{X},t)))$

Also $\frac{\partial}{\partial x_i} = \frac{\partial X_j}{\partial x_i}\frac{\partial}{\partial X_j} = K_{ji}\frac{\partial}{\partial X_j}$ (use summation convention)

So $\frac{\partial v}{\partial x_i} = K_{ji}\frac{\partial v}{\partial X_j}$

$$\nabla^2 v = \frac{\partial^2 v}{\partial x_i \partial x_i} = \frac{\partial}{\partial x_i}\left[K_{ji}\frac{\partial v}{\partial X_j}\right] = K_{ki}\frac{\partial}{\partial X_k}\left[K_{ji}\frac{\partial v}{\partial X_j}\right]$$

$$= K_{ki}K_{ji}\frac{\partial^2 v}{\partial X_j \partial X_k} + K_{ki}\frac{\partial K_{ji}}{\partial X_k}\frac{\partial v}{\partial X_j}$$

So the equation is $(v = v(\underline{X}, t))$, $\dfrac{\partial v}{\partial x_i} = K_{ji}\dfrac{\partial v}{\partial X_j}$

$$\varepsilon v_t = \varepsilon \frac{\partial x_i}{\partial t}\bigg|_{\underline{X}} K_{ji}\frac{\partial v}{\partial X_j} + \varepsilon^2\left[K_{ki}K_{ji}\frac{\partial^2 v}{\partial X_j \partial X_k} + K_{ki}\frac{\partial K_{ji}}{\partial X_k}\frac{\partial v}{\partial X_j}\right]$$

Note $K_{ki}\dfrac{\partial K_{ji}}{\partial X_k} = \dfrac{\partial X_k}{\partial x_i}\dfrac{\partial K_{ji}}{\partial X_k} = \dfrac{\partial K_{ji}}{\partial x_i}$

in particular $\dfrac{\partial K_{1i}}{\partial x_i} = \dfrac{\partial}{\partial x_i}\left[\dfrac{\partial X_1}{\partial x_i}\right]$

note also $\underline{\nabla} X_1$ is the normal to the interface

thus $\dfrac{\partial K_{1i}}{\partial x_i} = \underline{\nabla}\cdot[\underline{\nabla} X_1]$

We can assume (otherwise rescale X_1) that in fact $\underline{\nabla} X_1 = \underline{n}$. (unit normal)

then $n_i = K_{1i}$ and $\underline{K_{1i}K_{1i} = 1}$

$$\text{So } \varepsilon v_t = \varepsilon\frac{\partial x_i}{\partial t}\bigg|_{\underline{X}} K_{ji}\frac{\partial v}{\partial X_j} + \varepsilon^2 K_{ki}K_{ji}\frac{\partial^2 v}{\partial X_j \partial X_k} + \varepsilon^2\frac{\partial K_{ji}}{\partial x_i}\frac{\partial v}{\partial X_j}$$

Next we assume v changes rapidly with X_1, $X_1 = \varepsilon\xi$

then to leading order

$$0 = \underbrace{\frac{\partial x_i}{\partial t}\bigg|_{\underline{X}} K_{1i}}_{K_{1i} = n_i}\frac{\partial v}{\partial \xi} + \underbrace{K_{1i}K_{1i}}_{=1}\frac{\partial^2 v}{\partial \xi^2} + \varepsilon\underbrace{\frac{\partial K_{1i}}{\partial x_i}}_{\underline{\nabla}\cdot\underline{n}}\frac{\partial v}{\partial \xi} + f(v)$$

$$\Rightarrow 0 = v_{\xi\xi} + \Big[\underbrace{\underline{n}\cdot\frac{\partial \underline{x}}{\partial t}\Big|_{\underline{X}}}_{\substack{u_n \\ \text{normal velocity} \\ \text{of front}}} + \varepsilon\underline{\nabla}\cdot\underline{n}\Big]v_\xi + f(v)$$

and we know that the solution of this ~~eq~~ is $V(\xi)$ satisfying

$$V'' + cV' + f(V) = 0$$

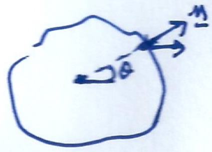
Therefore

$$\underbrace{\underline{n}\cdot\frac{\partial \underline{x}}{\partial t}}_{u_n} + \varepsilon \underline{\nabla}\cdot\underline{n} = c$$

(iii) $\underline{n}\cdot\frac{\partial \underline{x}}{\partial t}$ is front speed

$\underline{\nabla}\cdot\underline{n}$ is curvature

(b) the wave front is at $r = R(\theta, t)$



So $x = R\cos\theta$
$y = R\sin\theta$

$$\underline{n} \propto (dy, -dx)$$

$$\propto (R'\sin\theta + R\cos\theta, \; -R'\cos\theta + R\sin\theta)$$

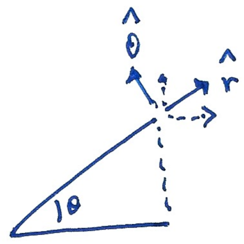
$$\text{so } \hat{n} = \frac{(R'\sin\theta + R\cos\theta, \; -R'\cos\theta + R\sin\theta)}{[R'^2\sin^2\theta + R^2\cos^2\theta + 2RR'\sin\theta\cos\theta + R'^2\cos^2\theta + R^2\sin^2\theta - 2RR'\sin\theta\cos\theta]^{1/2}}$$

$$= \frac{(R'\sin\theta + R\cos\theta, \; -R'\cos\theta + R\sin\theta)}{(R'^2 + R^2)^{1/2}}$$

$$\frac{\partial \underline{x}}{\partial t} = R_t(\cos\theta, \sin\theta)$$

$$\text{so } \underline{n}\cdot\frac{\partial \underline{x}}{\partial t} = \frac{R_t\,[R'\sin\theta\cos\theta + R\cos^2\theta - R'\sin\theta\cos\theta + R\sin^2\theta]}{(R^2 + R'^2)^{1/2}}$$

$$= \frac{RR_t}{(R^2 + R'^2)^{1/2}}$$



In cartesian coordinates

$$\underline{n} = \frac{(R'\sin\theta + R\cos\theta,\ -R'\cos\theta + R\sin\theta)}{(R^2+R'^2)^{1/2}}$$

we want this in cylindrical polars

note $\hat{\underline{r}} = \underline{i}\cos\theta + \underline{j}\sin\theta$

$\hat{\underline{\theta}} = -\underline{i}\sin\theta + \underline{j}\cos\theta$

so $\underline{i} = \hat{\underline{r}}\cos\theta - \hat{\underline{\theta}}\sin\theta$

$\underline{j} = \hat{\underline{r}}\sin\theta + \hat{\underline{\theta}}\cos\theta$

& so $\underline{n} = \frac{1}{(R^2+R'^2)^{1/2}}\Big[\underbrace{(R'\sin\theta + R\cos\theta)\cos\theta + (-R'\cos\theta + R\sin\theta)\sin\theta}_{\hat{r}},$

$\underbrace{-(R'\sin\theta + R\cos\theta)\sin\theta + (-R'\cos\theta + R\sin\theta)\cos\theta}_{\hat{\theta}}\Big]$

$$= \frac{1}{(R^2+R'^2)^{1/2}}\left[R, -R'\right]$$

$$\& \ \underline{\nabla}\cdot\underline{n} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{R}{(R^2+R'^2)^{1/2}}\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{-R'}{(R^2+R'^2)^{1/2}}\right]$$

at $r=R$, $$= \frac{1}{R}\left[\frac{R}{(R^2+R'^2)^{1/2}} - \frac{R''}{(R^2+R'^2)^{1/2}} + \frac{R'(RR'+R'R'')}{(R^2+R'^2)^{3/2}}\right]$$

$$= \frac{1}{R(R^2+R'^2)^{3/2}}\left[R(R^2+R'^2) - R''(R^2+\cancel{R'^2}) + RR'^2 + \cancel{R'^2R''}\right]$$

$$= \frac{1}{R(R^2+R'^2)^{3/2}}\left[R^3 + 2RR'^2 - R^2R''\right]$$

$$= \frac{R^2+2R'^2-RR''}{(R^2+R'^2)^{3/2}}$$

phew

So $$c = \frac{RR_t}{(R^2+R'^2)^{1/2}} + \frac{\varepsilon\,(R^2+2R'^2-RR'')}{(R^2+R'^2)^{3/2}}$$

or $$R_t = \frac{c\,(R^2+R'^2)^{1/2}}{R} - \frac{\varepsilon\,(R^2+2R'^2-RR'')}{R\,(R^2+R'^2)}$$ $\left(R' = R'(\theta) = \frac{\partial R}{\partial \theta}\right)$

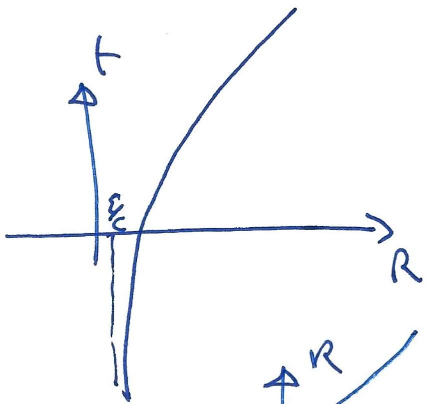
!!!

(ii) $R = R(t)$ $\Rightarrow \dot{R} = c - \frac{\varepsilon}{R} = \frac{cR-\varepsilon}{R}$

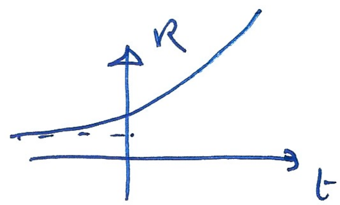
So $$\frac{R\dot{R}}{cR-\varepsilon} = 1$$

$$= \frac{R - \frac{\varepsilon}{c} + \frac{\varepsilon}{c}}{c\,(R-\frac{\varepsilon}{c})}$$

$$\Rightarrow ct = R + \frac{\varepsilon}{c}\ln\left(R-\frac{\varepsilon}{c}\right)$$



So I suppose curvature blocking refers to the fact that it doesn't work for small R (curvature too large)



iii, seems a bit long....

$r = R(t) + \rho \qquad \rho(\theta, t)$ small

$$\text{so} \quad \dot{R} + \rho_t = \frac{c\left(R^2 + 2R\rho \dots \dots\right)^{1/2}}{(R+\rho)} + \varepsilon \frac{\left[R\rho_{\theta\theta} \dots - R^2 - 2R\rho \dots\right]}{(R+\rho)(R^2 + 2R\rho \dots)}$$

$\times$ by $R+\rho$

$$R\dot{R} + R\rho_t + \dot{R}\rho + \dots = cR\left(1 + \frac{\cancel{2}\rho}{R} \dots\right) \cancel{+} - \varepsilon + \frac{\varepsilon}{R}\rho_{\theta\theta} \dots$$

$O(1)$ cancel so $(R\dot{R} = cR - \varepsilon)$

$$R\rho_t + \dot{R}\rho = \cancel{2}c\rho + \frac{\varepsilon}{R}\rho_{\theta\theta}$$

$\& \ \dot{R} = c - \frac{\varepsilon}{R} \qquad \text{So} \quad R\dot{\rho} = \frac{\varepsilon}{R}(\rho + \rho_{\theta\theta})$

& solutions are $\rho = \sum_m e^{im\theta} \rho_m(t)$ ($m \neq 0$ as we can take mean of ρ to be 0 w.l.o.g.)

$$\Rightarrow \quad R\dot{\rho}_m = \frac{\varepsilon}{R}(\rho_m - m^2\rho_m)$$
$$= -\frac{\varepsilon}{R}(m^2 - 1)\rho_m$$

so $|m| > 1$ decay

$|m| = 1$ neutrally stable apparently (modulo algebra)