C5.12

Math Physiol 2016 q2

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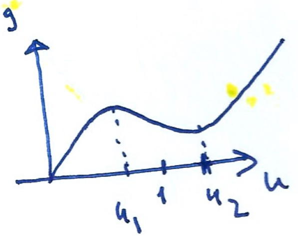
2,

$$u_t + \gamma v_t = 1 - u + \varepsilon u_{xx}$$

$$\varepsilon v_t = -v + g(u)$$



(a) v is the Ca^{2+}-sensitive store / concentration of Ca^{2+} in the cell (sarcoplasmic reticulum)

u is the cytosolic Ca^{2+} concentration in the cell.

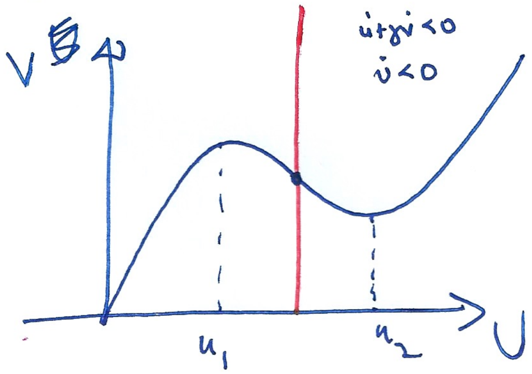
(b) $u = U(\xi)$ $v = V(\xi)$ $\xi = x + st$

$$\Rightarrow \quad s(U' + \gamma V') = 1 - U + \varepsilon U''$$

$$\varepsilon s V' = -V + g(U)$$

(U, V) phase plane.

Question a bit imprecise - periodic travelling waves? solitary travelling waves? ~~periodic~~ ~~[illegible]~~.



kinetics:

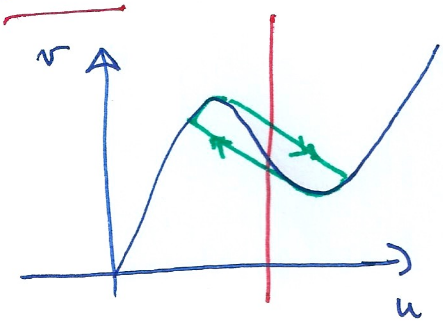
$$u_t + \gamma v_t = 1 - u$$

$$\varepsilon v_t = -v + g(u)$$

See below (p 3) for wave trajectory

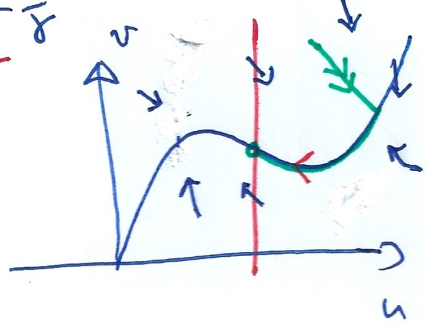
[Digression: whether the steady state is stable or unstable probably depends on whether $g' \gtrless -\frac{1}{\gamma}$ at the fixed point

if $\underline{g' < -\frac{1}{\gamma}}$ $\left(-g' \gtrsim \frac{1}{\gamma}\right)$



we get oscillations

if $\underline{g' > -\frac{1}{\gamma}}$



fixed point is stable

[Warning: see p2A for phase plane discussion]

Linearisation
$$\dot{u} = 1 - u - \frac{\gamma}{\varepsilon}\left[-v + g(u)\right]$$
$$\dot{v} = \frac{1}{\varepsilon}\left[-v + g(u)\right]$$

Near fixed pt community matrix is
$$M = \begin{pmatrix} -1 - \frac{\gamma}{\varepsilon} g' & \frac{\gamma}{\varepsilon} \\ \frac{g'}{\varepsilon} & -\frac{1}{\varepsilon} \end{pmatrix}$$

Phase plane indicates not saddle

fixed pt unstable iff $\operatorname{tr} M > 0$
$$\Rightarrow \quad -\frac{\gamma}{\varepsilon} g' - \frac{1}{\varepsilon} \gtrsim 0$$
$$\text{i.e.} \quad \underline{-g' \gtrsim \frac{1}{\gamma}}$$
]

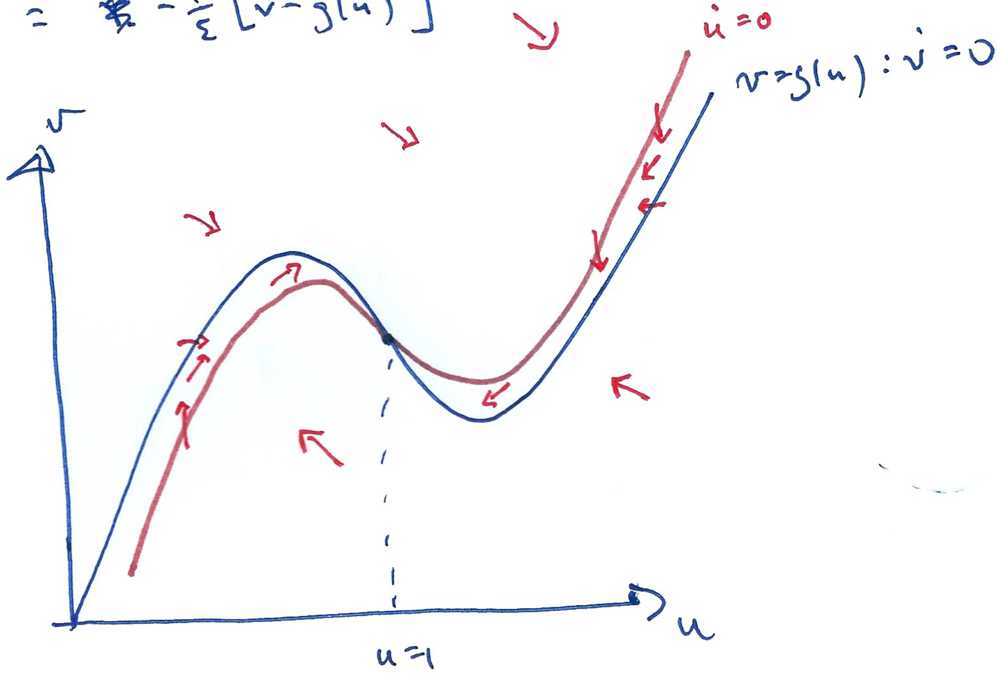
[Further digression and salutary warning:

If you look at the trajectory directions on p 2
it looks as if $\dot{u}=0$ on $v=g(u)$ instead of $\dot{v}$!?

In more detail:

$$\dot{u} = 1-u + \frac{\gamma}{\varepsilon}[v-g(u)]$$
$$\dot{v} = -\frac{1}{\varepsilon}[v-g(u)]$$

Therefore



actual nullclines for u & v are

$\dot{v}=0 \quad v=g(u)$ (blue)

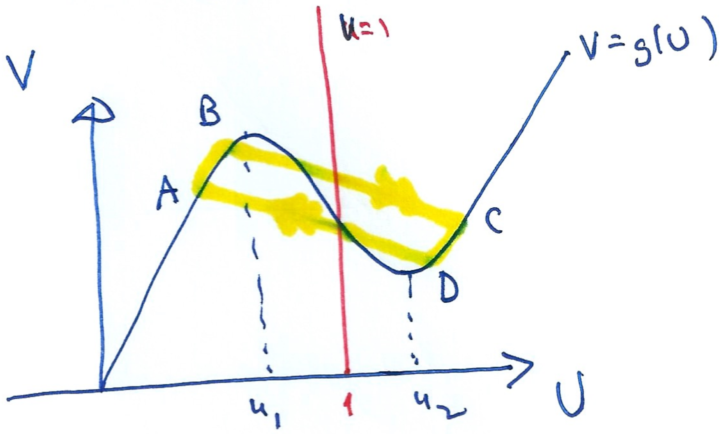
$\dot{u}=0 \quad v=g(u)+\frac{\varepsilon}{\gamma}(u-1)$ (brown)

and trajectory directions as shown in red

Warning: do not be overly simplistic with phase plane
– rely on $\varepsilon \ll 1$]

The diagram in the question is terrible!
But appears to indicate a solitary wave - but the system is not excitable, so let us assume periodic travelling waves

Then ($\sim$ Fitzhugh-Nagumo) I guess



(BC & AD not necessarily straight)

we don't know what U_B or U_A are

(c) Fast: $\xi = \varepsilon X$

$$\Rightarrow \quad s(U' + \gamma V') = \varepsilon[1-U] + U''$$

$$sV' = -V + g(U)$$

$$\cancel{\Rightarrow sU' + \gamma[-V+g(U)] \simeq U''}$$

1st is $\quad s(U' + \gamma V') \simeq U''$

$$\Rightarrow \quad s\left[U - U_B + \gamma(V - V_B)\right] \simeq U' \qquad (U \to U_B,\ V \to V_B \text{ as } X \to -\infty)$$

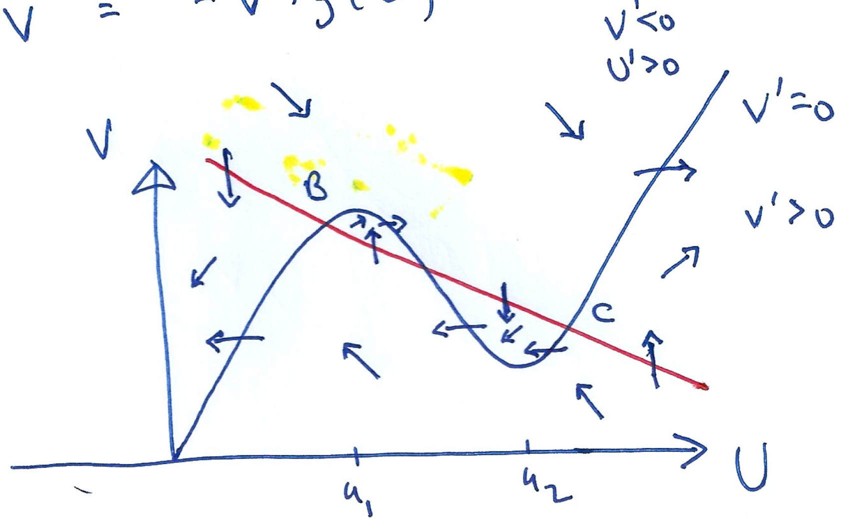
as required

(i) Suitable choice of γ !! We already need to know this! It is $g'(1) < -\frac{1}{\gamma}$ (so oscillatory kinetics)

$$U' = s\left[U - U_B + \gamma(V - V_B)\right]$$

$$sV' = -V + g(U)$$

$s > 0$



Phase portrait : nullclines (above), directions (above)

Whence it is clear that B & C are saddles

& intermediate fixed point is node or spiral

stability depends on tr M, $M = \begin{pmatrix} s & \gamma s \\ g' & -1 \end{pmatrix}$

$\mathrm{tr} M = s - 1$

unstable if $s > 1$

stable if $s < 1$

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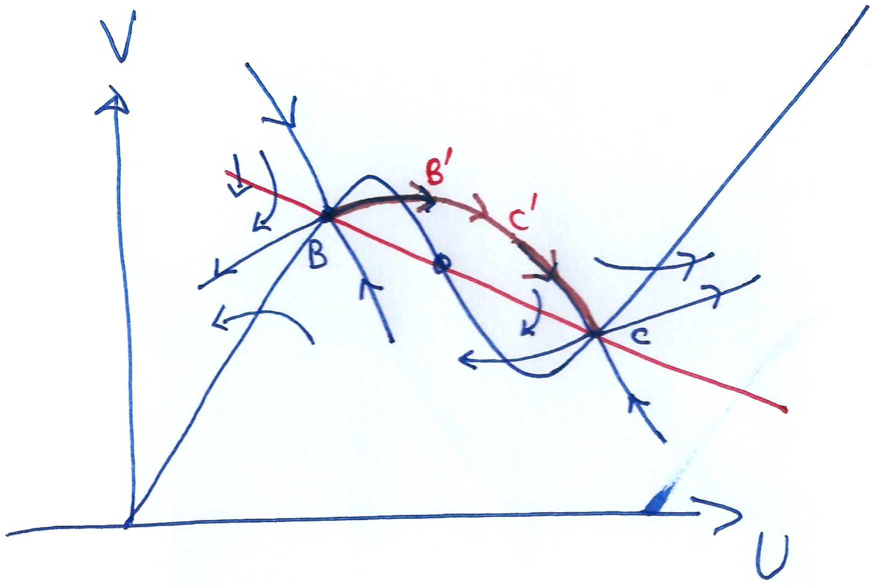
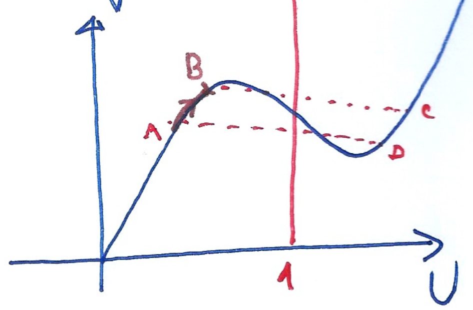


ii To get from B to C the unstable separatrix BB' from B must join to the stable separatrix C'C to C. In general this will not occur. Given B, we choose s to make it happen.

(d) Back to ξ : $s(U' + \gamma V') = 1 - U + \varepsilon U''$

$$\varepsilon s V' = -V + g(U)$$



$\Rightarrow$ $V = g(U)$

$s(U' + \gamma V') = 1 - U$

$\Rightarrow$ $s\left[1 + \gamma g'(U)\right]U' = 1 - U$

"Hence show..." – imprecise. I think what is meant is that at B, $U_B < 1$, $U' > 0$,

$\Rightarrow$ $1 + \gamma g'(U_B) > 0$ so U_B is less than the value it obtains in the limit cycle (which has $\gamma g' = -1$)

[Note: U_B is not constrained: given U_B, choose s; on the way back, select D to connect D to A. So there is a one-parameter family of periodic travelling waves]