C5.12 Math Physiol 2017 q1 answer

C1

1. $$\varepsilon \dot{v} = I^* + f(v) - w$$
$$\dot{w} = \gamma v - w$$
$$f = -v(a-v)(1-v) \qquad 0<a<1$$

(a) v intracellular electric potential

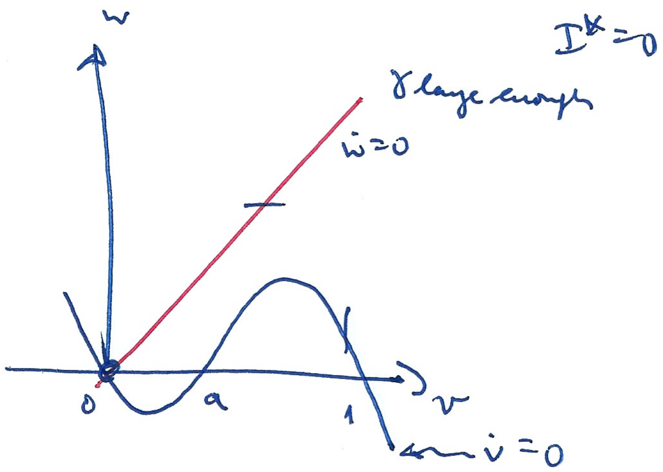
w gate variable

I^* applied current

$-f(v)+w$ ionic current out

Start with Hodgkin–Huxley, 3 gate variables m,h,n & telegraph equation for potential (via I_H & I_L) : $\tau_m \ll 1 \Rightarrow m$ equilibrium, $n+h=$ constant.
$\Rightarrow$ FN reduction ...

(b) $I^*=0$



Evidently one equilibrium $(0,0)$ for γ large

Linearise $$\begin{pmatrix}\dot{v}\\ \dot{w}\end{pmatrix} \simeq \underbrace{\begin{pmatrix}\frac{f'}{\varepsilon} & -\frac{1}{\varepsilon}\\ \gamma & -1\end{pmatrix}}_{M}\begin{pmatrix}v\\ w\end{pmatrix}$$

$\operatorname{tr} M = \frac{f'}{\varepsilon} - 1 < 0$ at $(0,0)$, $\det M = -\frac{f'}{\varepsilon} + \frac{\gamma}{\varepsilon} > 0 \Rightarrow$ stable

$t \sim 1$: $w \simeq f(v)$

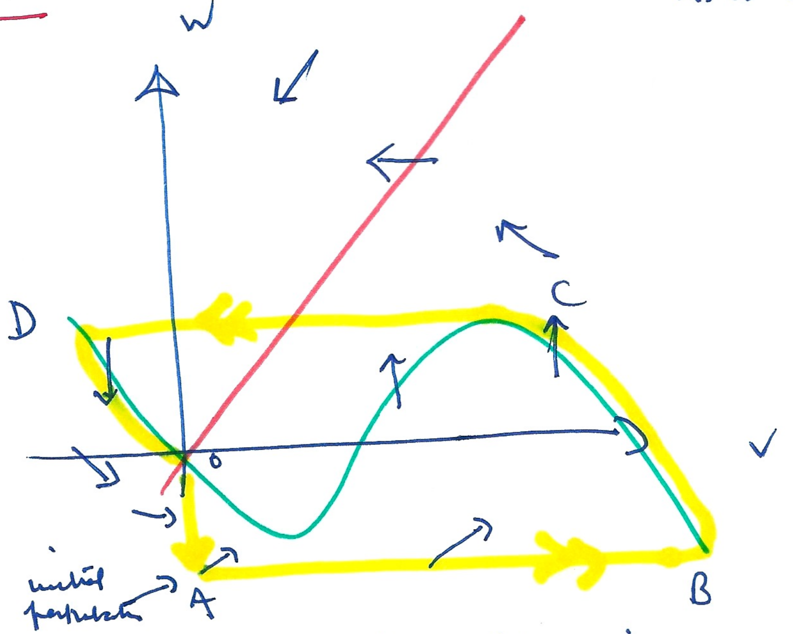
$\dot{w} = \gamma v - w$

$t \sim \varepsilon$, $t = \varepsilon T$ $v' = f(v) - w$

$w' = 0$

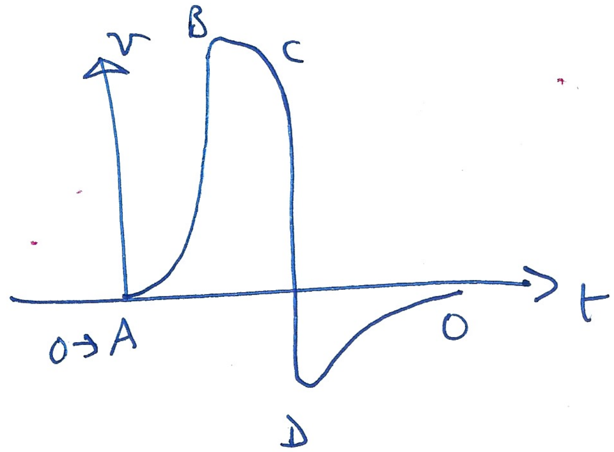
Phase plane

As $w \to \infty$ & $\dot{v} < 0$
$\dot{w} < 0$



trajectories cycle round stable origin

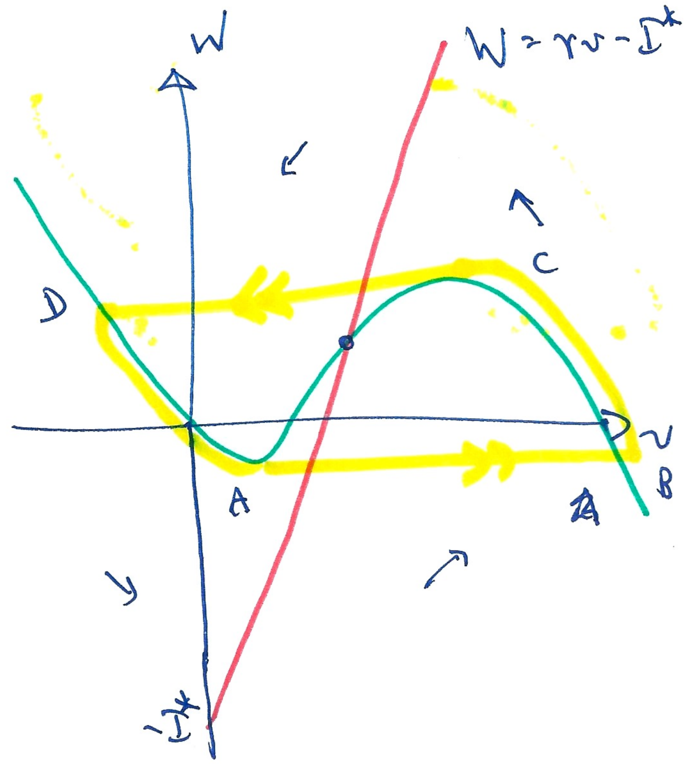
threshold : eg applies as the $I^{*} > 0$ for short time $\longleftrightarrow$ lower $w \Rightarrow$ trajectory as shown : fast phases with $\twoheadrightarrow$



(e) $I^* = 0$, $W = w - I^*$

$$\varepsilon \dot{v} = f(v) - W$$

$$\dot{W} = \gamma v - I^* - W$$



$$p \approx \int_B^C dt + \int_D^A dt$$

$$\approx \int_B^C \frac{dW}{\dot{W}} + \int_D^A \frac{dW}{\dot{W}} \qquad \text{along } W = f(v)$$

$$= \int_B^C \frac{dW}{\gamma v - I^* - W} + \int_D^A \frac{dW}{\gamma v - I^* - W}$$

$$= \int_B^C \frac{f'(v)\,dv}{\gamma v - I^* - f(v)} + \int_D^A \frac{f'(v)\,dv}{\gamma v - I^* - W}$$

~~Hmm~~ oh question leaves as W, how odd

$$P \simeq \int_{W_B}^{W_C} \frac{dW}{\gamma v - I^* - W} + \int_{W_D}^{W_A} \frac{dW}{\gamma v - I^* - W}$$

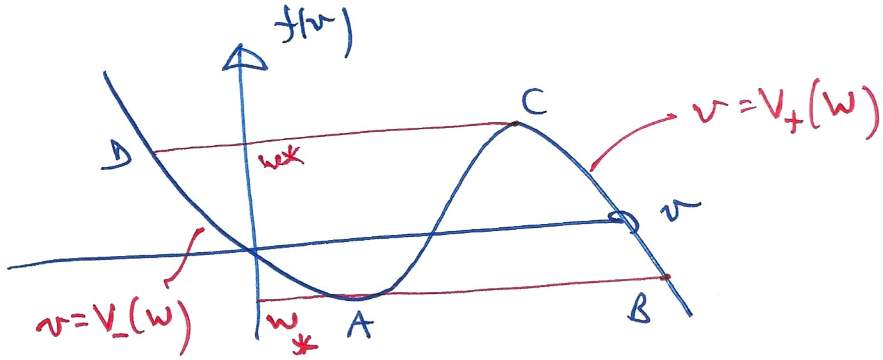
On BC $W < \gamma v - I^* - W$, denote $v = V_+(W)$ on BC

on DA $W > \gamma v - I^* - W$ denote $v = V_-(W)$ on AD

(see figure)

Also let $W = f(v) = w_*$ at A & B

$W = w^*$ at C & D

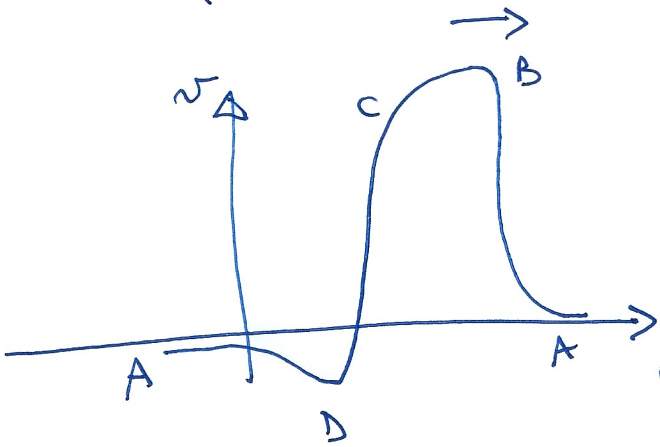


So $$P \simeq \int_{w_*}^{w^*} \frac{dW}{\gamma V_+(W) - I^* - W} + \int_{w^*}^{w_*} \frac{dW}{\gamma V_-(W) - I^* - W}$$

$$= \int_{w_*}^{w^*} \left[\frac{1}{\gamma V_+(W) - I^* - W} - \underbrace{\frac{1}{\gamma V_-(W) - I^* - W}}_{\text{note this is } > 0} \right] dW$$

(d) $\varepsilon v_t = f(v) - w + \varepsilon^2 v_{xx}$

$w_t = \gamma v - w$



$y = x - ct \qquad c > 0$

(i) $-\varepsilon c v' = f(v) - w + \varepsilon^2 v''$ $\qquad v(Y), w(Y)$

$-cw' = \gamma v - w$ $\qquad$ bc's $v, w \to 0$ at $\pm\infty$.

(ii) AB fast ~~time~~ $y = \varepsilon Y$

$\Rightarrow \quad -cv' = f(v) - w + v''$

$-cw' = \varepsilon(\gamma v - w) \Rightarrow w \simeq 0$

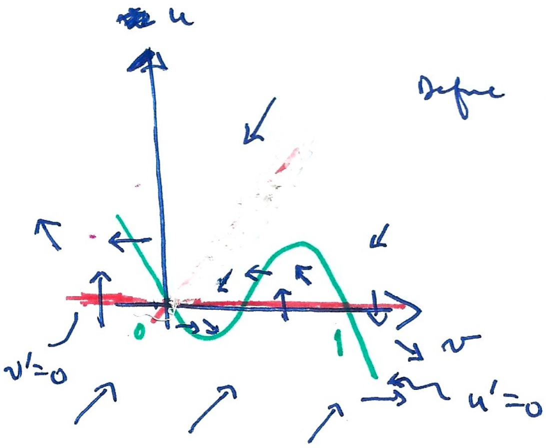
$\Rightarrow v'' + cv' + f(v) = 0$ in AB

$Y \to \infty \quad v \to 0$

$Y \to -\infty \quad v \to v_B$ (where $f(v_B) = 0$ so $v_B = 1$

ie $v \to 1$ (not a).

(iii)



Define ~~then~~ $v' = -u$

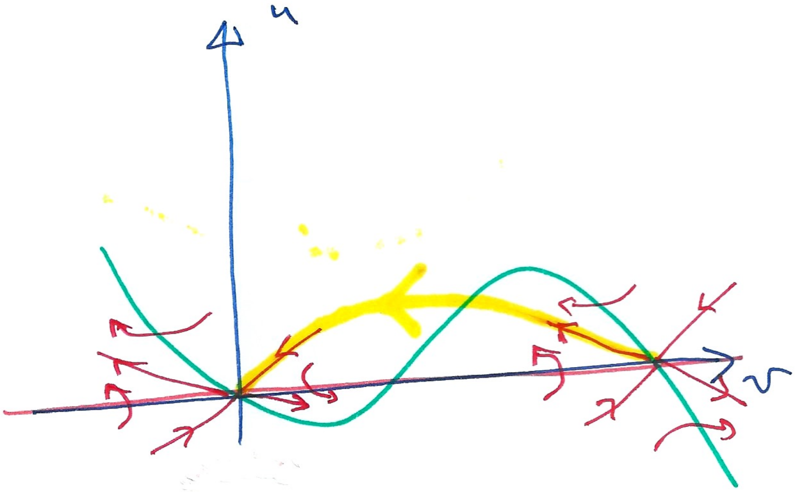
then $u' = f(v) - cu$

At large u $v' < 0$ $u' < 0$

$\Rightarrow$ trajectories as shown

Clearly $(0,0)$ & $(1,0)$ are saddles, $(a,0)$ a node or spiral

To connect $(1,0)$ $(y \to -\infty)$ to $(0,0)$ $(y \to \infty)$ we need to connect the unstable separatrix from $(1,0)$ in $v<1$ to the stable separatrix to $(0,0)$ in $v>0$:



This requires a choice of c to make them join.

$$v'' + cv' + f(v) = 0$$

So $(\times v')$
$$\tfrac{1}{2} v'^2 \Big|_{-\infty}^{\infty} + c \int_{-\infty}^{\infty} v'^2 \, dy + \int_1^0 f(v)\, dv = 0$$

$$\Rightarrow \quad c = \frac{\int_0^1 f(v)\, dv}{\int_{-\infty}^{\infty} v'^2 \, dy}$$

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which requires $\int_0^1 f(v)\, dv > 0$

viz $a < \frac{1}{2}$