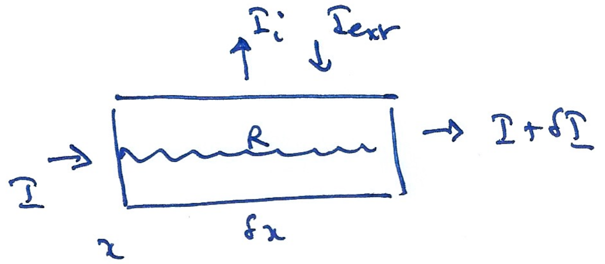
C5.12

Math Physiol 2018 q1

(a)



Conservation of charge (density) $Q = CV$

$$\frac{\partial}{\partial t} Q\delta x = I\Big|_{x+\delta x}^{x} + (I_{ext} - I_i)\,\delta x$$

$$\Rightarrow \quad C\frac{\partial V}{\partial t} = -\frac{\partial I}{\partial x} + I_{ext} - I_i$$

Also Ohm's law with Resistance R / unit length

$$-\delta V = R\delta x\, I$$

$$\Rightarrow \quad -\frac{\partial V}{\partial x} = RI$$

C capacitance (/unit length)
R resistance (/unit length

(b) $\quad I_i = g_1(V-V_1)n + g_2(V-V_2)(1-n)m \qquad V_1 < V_2$

$$\tau \dot{n} = n_\infty(V) - n$$

(i) steady state $I_i = 0 \Rightarrow V = V_0$, $n = n_\infty(V_0) = n_0$ say

$$\Rightarrow 0 = g_1(V_0 - V_1)n_0 + g_2(V_0 - V_2)(1-n_0)m \qquad \text{or } m = m(V_0)?$$

$$\Rightarrow \quad V_0 = \frac{g_1 n_0 V_1 + g_2(1-n_0)m V_2}{g_1 n_0 + g_2(1-n_0)m} \in (V_1, V_2)$$

since $g_1 n_0 > 0$
$g_2(1-n_0)m > 0$

ii

non-d $\quad t \sim \tau \quad x \sim l \quad V = V_0 + V^* v$

$$C V_t = \frac{1}{R} V_{xx} + I_{ext} - I_i$$

& $I_i = g_1 [V_0 - V_1 + V^* v] n + g_2 [V_0 - V_2 + V^* v](1-n)m$

$$= g_1 V^* \left[\frac{V_0 - V_1}{V^*} + v\right] n + g_2 V^* \left[v - \frac{V_2 - V_0}{V^*}\right](1-n)m$$

$$= g_2 V^* I_i^* \, ,$$

$$I_i^* = \frac{g_1}{g_2}\left[\frac{V_0 - V_1}{V^*} + v\right] n + \left[v - \frac{V_2 - V_0}{V^*}\right](1-n)m$$

choose $\quad V^* = V_2 - V_0 > 0$

$$\Rightarrow I_i^* = \gamma (v + v_1) n + (v-1)(1-n)m$$

$$\gamma = \frac{g_1}{g_2}, \quad v_1 = \frac{V_0 - V_1}{V_2 - V_0} > 0$$

and $\quad \dfrac{C V^*}{\tau} v_t = \dfrac{V^*}{R l^2} v_{xx} + g_2 V^* (I_{ext}^* - I_i^*)$

where also $\quad I_{ext} = g_2 V^* I_{ext}^*$

& $\quad n' = n_0 - n$

Thus define $\quad \varepsilon = \dfrac{C V^*}{\tau g_2 V^*} = \dfrac{C}{\tau g_2}$

& $\varepsilon^2 = \dfrac{1}{g_2 R l^2}$ ie $\quad l = \dfrac{\tau g_2}{C \sqrt{g_2 R}} = \dfrac{\tau}{C}\sqrt{\dfrac{g_2}{R}}$

(e) $\varepsilon v_t = I^*_{ext} - I^*_i + \varepsilon^2 v_{xx}$

$n_t = n_\infty - n$

$I_i^* = g(n,v) = \gamma(v+v_1)n + (v-1)(1-n)m$

$n = h(v) \iff g(n,v) = 0$

explicitly $\left[\gamma(v+v_1) - (v-1)m\right]n + (v-1)m = 0$

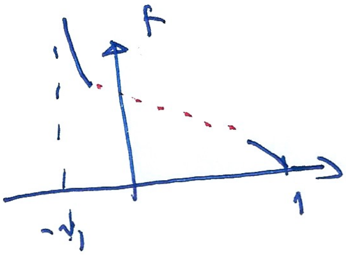
$$n = \frac{(v-1)m(v)}{(v-1)m(v) - \gamma(v+v_1)} = h(v) \qquad m>0, \; m'>0$$

equilibrium: $v=0$, $n = n_\infty(0)$

Consider the function $f(v) = \dfrac{(1-v)m(v)}{\gamma(v+v_1)}$ (so $h = \dfrac{1}{f+1}$)

For $-v_1 < v < 1$

$f > 0$

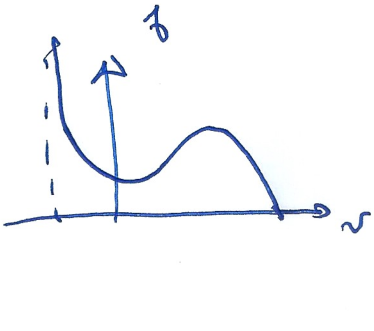


$f \to \infty$ at $-v_1$
$f \to 0$ at 1

so monotonically decreasing is possible

but clearly f will now be non-monotonic if

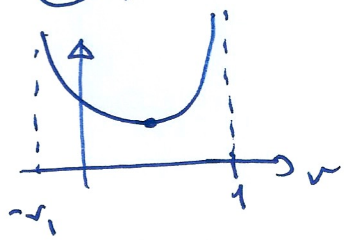
m' is large enough



....

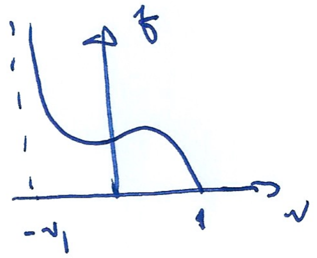
Explicitly $\frac{f'}{f} = \frac{-1}{1-v} + \frac{m'}{m} - \frac{1}{v+v_1}$

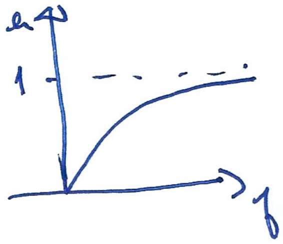
$$= \frac{m'}{m} - \underbrace{\left[\frac{1}{1-v} + \frac{1}{v+v_1}\right]}$$



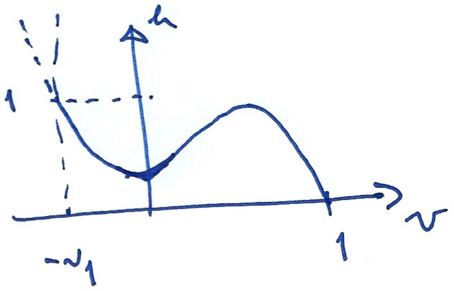
>0 if $\frac{m'}{m}$ larger than $\min_{-v_1<v<1}\left[\frac{1}{1-v} + \frac{1}{v+v_1}\right]$

then





$\Rightarrow$



(ii) obvious (already done). $h'(v) = h'(f)\, f'(v)$

$$= \frac{f}{(f+1)^2}\left[\frac{m'}{m} - \left\{\frac{1}{1-v} + \frac{1}{v+v_1}\right\}\right]$$

$\to -\infty$ as $v \to 1$

$h < 0$ as $v \to v_1$, $f \to \infty$

$$\left(h' \approx \frac{1}{f} \cdot -\frac{1}{(v+v_1)} \approx -\frac{\gamma}{(1+v_1)\, m(-v_1)}\right)$$

$m = \exp[k(v-1)]$

$= \frac{m'}{m} = k$

non-monotonic if $k > \min\left[\frac{1}{1-v} + \frac{1}{v+v_1}\right]$

which is at $+\frac{1}{(1-v)^2} = \frac{1}{(v+v_1)^2}$

ie $1-v = v+v_1$

$v = \frac{1-v_1}{2}$

min is $\left[\frac{1}{1-\left(\frac{1-v_1}{2}\right)} + \frac{1}{\frac{1-v_1}{2}+v_1}\right]$

$= \frac{4}{1+v_1}$

So if $k > \frac{4}{1+v_1}$

(iii) $n_\infty(v)$ increasing

~~At $v=0$, $f = \frac{m(0)}{\gamma v_1}$~~

$h'(v) = \frac{f}{(f+1)^2}\left[k - \left\{\frac{1}{1-v} + \frac{1}{v+v_1}\right\}\right]$

$h'(0) > 0$ if $[f(0) > 0$ as $m(0) > 0]$

$k > 1 + \frac{1}{v_1}$

[Note: question suggests $h'(0)>0$ as below. See p7 for $h'(0)<0$]

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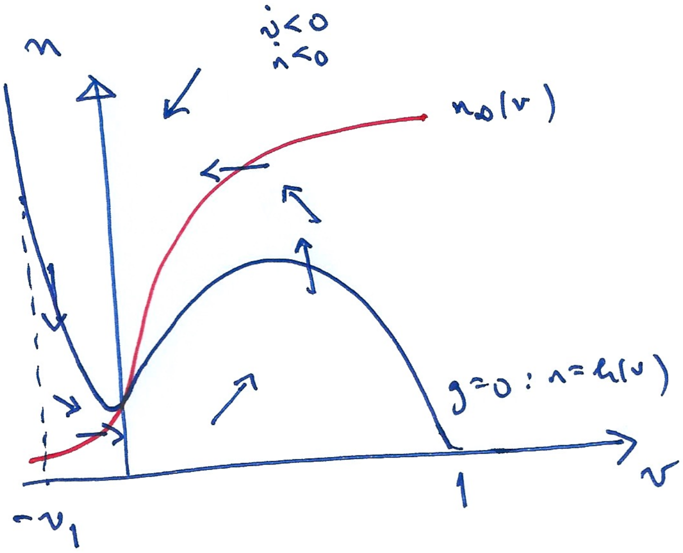
$\varepsilon \dot{v} = -g(n,v)$

$\dot{n} = n_\infty - n$

Note for $-v_1<v<1$, $n>0$

$g \to \infty$ as $n \to \infty$

so $\dot{v}<0$ for $n>h(v)$



$\downarrow$

trajectory directions as shown

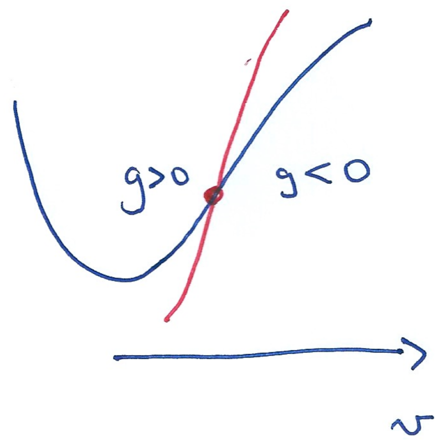
$\Rightarrow$ fixed point is node or spiral

$\Rightarrow$ stable if $\operatorname{tr} M<0$ where $M = \begin{pmatrix} -\frac{1}{\varepsilon}g_v & -\frac{1}{\varepsilon}g_n \\ n_\infty' & -1 \end{pmatrix}$

ie $\operatorname{tr} M = -\frac{1}{\varepsilon}g_v - 1$

<0 if $g_v > -\varepsilon$

Now if $h'(0)>0$



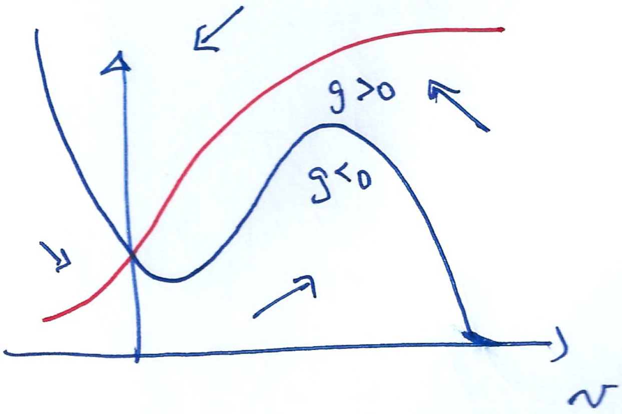
$\Rightarrow g_v<0$

and the steady state is unstable !?

[The question is not specific, but the implication to me is that we should have $h'(0)<0$]

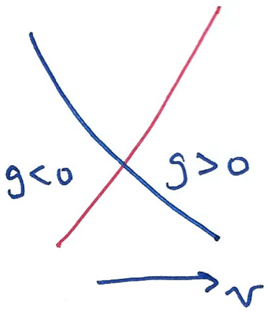
[I think the question ought to read:

Determine the conditions such that $h'(0) < 0$. Assuming this, we...]



Steady state is stable if $g_v > -\varepsilon$

i.e. if $g_v \gtrsim 0$ $(\varepsilon \ll 1)$



$\Rightarrow g_v > 0$

Threshold phenomenon: increase v