Mathematical physiology

PROBLEM SHEET 3.

1. The CICR model is given by

$$\begin{aligned} u_t + \gamma v_t &= \mu - u, \\ \varepsilon v_t &= f(u,v), \end{aligned}$$

where
$$f = \beta\left(\frac{u^n}{1+u^n}\right) - \left(\frac{v^m}{1+v^m}\right)\left(\frac{u^p}{\alpha^p+u^p}\right) - \delta v.$$

Show that it has a unique steady state (with $u, v > 0$). Show that it is oscillatorily unstable if
$$\varepsilon - f_v < -\gamma f_u$$
at the fixed point, and deduce that if $g(u)$ is defined by $f[u, g(u)] = 0$, and $\varepsilon \ll 1$, then this criterion is approximately

$$g'(\mu) < -1/\gamma.$$

Deduce from the form of the graph of $g(u)$ that periodic solutions will exist in a range $\mu_- < \mu < \mu_+$.

What might the instability region be in the (μ, δ) plane?

2. The dimensionless two-pool model of CICR,

$$\begin{aligned} u_t &= \mu - u - \frac{\gamma f(u,v)}{\varepsilon}, \\ \varepsilon v_t &= f(u,v), \end{aligned}$$

is considered in a one-dimensional spatial domain. Explain why the model may be modified by a diffusion term in u but not in v, and explain also why the natural length scale to choose is such that the scaled term is εu_{xx}.

Suppose that $f(u,v) = 0$ defines a function $v = g(u)$ with $g(0) = 0$, that g first increases to a large maximum, decreases to a positive minimum, and then increases to an asymptote as $u \to \infty$, and that $g' < -1/\gamma$ for $\mu_- < u < \mu_+$, where $\mu_+ > \mu_- > 0$. Use phase plane analysis to show plausibly that periodic travelling wave trains will exist for $\mu_- < \mu < \mu_+$, where also $g'(\mu_\pm) = -1/\gamma$ (assuming $\min g' < -1/\gamma$).

3. The equation
$$v_t = f(v) + \nabla^2 v$$

admits a unique travelling wave solution in one dimension of the form

$$v = V(\xi), \quad \xi = x - ct, \quad c > 0, \quad V(-\infty) = 1, \quad V(\infty) = 0.$$

Write down the ordinary differential equation satisfied by V.

Suppose now that we seek a solution in two dimensions which is slowly varying in the direction transverse to the direction of propagation. By seeking a solution in the form $v = V[\psi(\mathbf{x}, t)]$, show that

$$V'(\psi_t - \nabla^2\psi) = f(V) + V''|\boldsymbol{\nabla}\psi|^2,$$

where $V' = dV/d\psi$.

Let ξ be a (curvilinear) coordinate orthogonal to the wavefront, taken to be $\psi(\mathbf{x}, t) = 0$. Show that

$$\frac{\partial\psi}{\partial\xi} = |\boldsymbol{\nabla}\psi|,$$

that the normal velocity of the interface is

$$v_n = -\frac{\psi_t}{|\boldsymbol{\nabla}\psi|},$$

and that the curvature is

$$\boldsymbol{\nabla}.\,\mathbf{n} = \frac{1}{|\boldsymbol{\nabla}\psi|}\left[\nabla^2\psi - \frac{\partial|\boldsymbol{\nabla}\psi|}{\partial n}\right],$$

where $\mathbf{n}$ is the unit normal to the front. Hence show that

$$V_{\xi\xi} + \frac{V_\xi}{|\boldsymbol{\nabla}\psi|}\left\{-\frac{\partial|\boldsymbol{\nabla}\psi|}{\partial\xi} + \nabla^2\psi - \psi_t\right\} + f(V) = 0,$$

and deduce that

$$v_n = c - \boldsymbol{\nabla}.\,\mathbf{n}.$$

Find a solution of this equation which describes a target wave. [*Hint: seek a solution* $\psi = f(r) - ct$.]

4. Describe the sequence of events which occurs in the human circulatory system during a single heart beat. Your description should include a schematic illustration of the circulatory system, how filling and emptying of the atria and ventricles is effected by valve opening and closing, and how this affects the pressure and volume of the left ventricle.

   What is meant by *stroke volume* and *heart rate*? How does the cardiac output depend on these?

   A simple model of the circulation consists of a (left) ventricle (with mitral and aortic valves), arteries, veins and capillaries. Show that a simple compartment model for this system which describes the volumes of the arteries, veins and ventricle can be written in the form

$$\dot{V}_a = Q_+ - Q_c,$$

$$\dot{V}_v = Q_c - Q_-,$$

$$\dot{V}_{\rm LV} = Q_- - Q_+,$$

and describe the meaning of the variables. What assumption is made about the capillary volume in writing these equations? How should the variables Q_k be determined in terms of arterial, venous, and ventricular pressures?

5. Picard's theorem states that a holomorphic function $f(z)$ having an isolated essential singularity at $z = z_0$ takes on every possible complex value in any neighbourhood of z_0, with at most one exception. Use this to show that the equation for σ,

$$\sigma = -\beta - \gamma e^{-\sigma},$$

where β and γ are positive constants, has an infinite number of complex roots in a neighbourhood of ∞.

Show that if $\sigma \to \infty$, then also $\operatorname{Re}\sigma \to -\infty$.

Show that the complex roots vary continuously with γ (for example show that $\partial\sigma/\partial\gamma$ exists for complex σ).

Show that $\operatorname{Re}\sigma < 0$ for all roots if γ is sufficiently small.

Deduce that instability occurs for $\gamma > \gamma_c$, where

$$\gamma_c = \frac{\Omega}{\sin\Omega},$$

and Ω is the smallest (positive) root of

$$\tan\Omega = -\frac{\Omega}{\beta}.$$

Use `Maple` or some other graphical software to plot γ_c as a function of β.