Problem sheet 2

1. **Carbon cycles** Consider the dimensionless model from lectures for the evolution of albedo and partial pressure of atmospheric CO_2,

$$\dot{a} = f(a,p) = B(\Theta) - a,$$
$$\dot{p} = g(a,p) = \alpha(1 - wp^{\mu}e^{\Theta}),$$

where $\Theta(a,p) = \frac{q(1-a)-1}{\nu} + \lambda p$, and $B(\theta)$ is a monotonic function decreasing from a_+ to a_-. Here, μ, α, ν, λ, w. and q are all constant parameters.

(i) Show that the p nullcline, $a = G(p)$ say, is a monotonically increasing function of p, and that the a nullcline, $a = F(p)$, is a monotonically decreasing function if $-B'(\theta) < \nu/q$ for all θ, but is multivalued if $-B'(\theta) > \nu/q$ for some range of θ.

(ii) Now suppose that the a nullcline is indeed multivalued, but that there is always a unique steady state, which may lie on the lower, intermediate, or upper branch depending on the value of w. Sketch the nullclines for each of these cases. By considering the signs of the partial derivatives of $f(a,p)$ and $g(a,p)$ (but without detailed calculation), show that steady states on the upper or lower branch are stable, but the intermediate state is unstable if α is small enough. How would you expect the solutions to behave if $\alpha \ll 1$?

2. **Ocean carbon** A model for the global climate is

$$c\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{1}{4}Q(1-a) - \sigma\gamma(p)T^4, \qquad t_i\frac{\mathrm{d}a}{\mathrm{d}t} = a_0(T) - a,$$

$$\frac{A_E M_{CO_2}}{gM_a}\frac{\mathrm{d}p}{\mathrm{d}t} = v - h(p - p_s), \qquad \rho_O V_O\frac{\mathrm{d}C}{\mathrm{d}t} = \frac{h(p-p_s)}{M_{CO_2}} - bC,$$

where $p_s = C/K$ is the effective partial pressure of CO_2 in the ocean (*i.e.* the partial pressure of a gas in equilibrium with the water).

(i) Briefly explain the meaning of the terms in this model and the physical principles on which it is based.

(ii) Estimate the timescales involved using the parameter values listed below. Hence show that a suitable quasi-steady approximation of the model is

$$t_i\dot{a} = a_0(T) - a, \qquad t_C\dot{C} = C_v - C,$$

where $t_C = \rho_O V_O/b$, $C_v = v/M_{CO_2}b$, and where

$$p \approx \frac{C}{K} + \frac{v}{h}, \qquad T \approx \left(\frac{Q(1-a)}{4\sigma\gamma(p)}\right)^{1/4}.$$

(iii) Assuming that the ocean was in equilibrium with pre-industrial emissions, infer the value of those emissions (*i.e.* v) given the present day value of $C \approx 2 \times 10^{-3}$ mol kg^{-1}, and estimate the pre-industrial value of atmospheric CO_2 given by p.

(iv) Suppose the present-day emissions $v \approx 30 \times 10^{12}$ kg y^{-1} are maintained indefinitely. Use the model to show that on a timescale of centuries p will reach an approximate equilibrium and find its value. Show that thereafter p will continue to increase, on a timescale of millennia. What is the eventual value of p?

Parameter values: $c = 10^7$ J m^{-2} K^{-1}, $Q = 1370$ W m^{-2}, $\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4}, $t_i = 10^4$ y, $A_E = 5.1 \times 10^{14}$ m^2, $g = 9.8$ m s^{-2}, $M_{CO_2} = 44 \times 10^{-3}$ kg mol^{-1}, $M_a = 29 \times 10^{-3}$ kg mol^{-1}, $h = 0.73 \times 10^{12}$ kg y^{-1} Pa^{-1}, $\rho_O = 10^3$ kg m^{-3}, $V_O = 1.35 \times 10^{18}$ m^3, $b = 0.83 \times 10^{16}$ kg y^{-1}, $K = 7.1 \times 10^{-5}$ mol kg^{-1} Pa^{-1}.

3. **River cross-sections**

(i) Calculate the relationship between hydraulic radius R and cross-sectional area A for (i) a rectangular cross-section of fixed width w (you may assume the width is much wider than the depth), and (ii) a triangular cross-section with transverse slope angle β.

(ii) Use Manning's law, $u = R^{2/3}S^{1/2}/n$, to derive the equation

$$\frac{\partial A}{\partial t} + cA^m \frac{\partial A}{\partial x} = E,$$

for the cross-sectional area of a river, where E is a prescribed source term, giving explicit formulas for c and m in each of case (i) and (ii).

(iii) What initial and boundary conditions you would expect to apply to this equation? Why would this equation not be able to describe the behaviour of a tidal river such as the Thames in London?

(iv) For the case $E = 0$, with $A(x, 0) = A_0(x)$ on $-\infty < x < \infty$, find an implicit solution to the equation. Show that if $A_0'(x) < 0$ for some x then a shock will form, and find expressions for the time and location at which the shock first forms in terms of $A_0(x)$.

4. **Overland flow** Overland flow on a hill slope is described by the equation

$$\frac{\partial h}{\partial t} + ch^m \frac{\partial h}{\partial x} = E,$$

where $E = P - I$ is the excess rainfall rate, being the difference between precipitation and infiltration rates. The equation is to be solved in $x > 0$, and the initial and boundary condition are

$$h = 0 \quad \text{at} \quad t = 0, x > 0 \quad \text{and} \quad x = 0, t > 0.$$

(i) First consider the case of constant $E > 0$. Solve the equation and sketch the solution at various times.

(ii) Next, consider the case where $E(t)$ is time dependent, such that $E \geq 0$ for $0 \leq t \leq t_*$, and $E < 0$ for $t > t_*$. Find an implicit expression for the solution in terms of integrals of E for $t \leq t_*$.

(iii) For $t > t_*$, a drying front moves down the slope (behind which $h = 0$). Determine the position of the front $x_d(t)$ as a function of time and hence find the complete solution for all $t > 0$.