Sheet 1

1. <u>Lapse Rates</u> $\rho c_p \frac{dT}{dz} - \frac{dp}{dz} = 0$ $\quad \frac{dp}{dz} = -\rho g$ $\quad p = \frac{\rho R T}{M_a}$ with $T = T_s$, $p = p_s$ at $z = 0$.

Combining $\Rightarrow \rho c_p \frac{dT}{dz} = -\rho g \Rightarrow \frac{dT}{dz} = -\frac{g}{c_p} \Rightarrow \boxed{T = T_s - \frac{g}{c_p} z}$ (note $\frac{g}{c_p} \approx 10 \text{ K km}^{-1}$)

Also $\frac{dp}{dz} = -\rho g = -\frac{M_a g}{RT} p \Rightarrow \frac{1}{p}\frac{dp}{dz} = \frac{M_a g}{R} \frac{1}{T_s - \frac{g}{c_p} z}$.

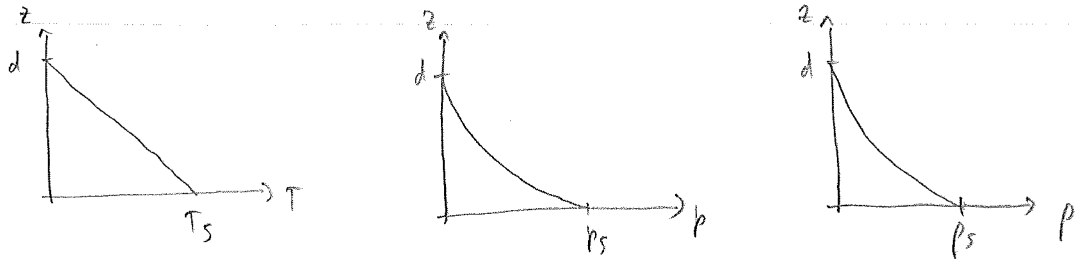
$$\Rightarrow \ln\left(\frac{p}{p_s}\right) = \frac{M_a c_p}{R} \ln\left(\frac{T_s - \frac{g}{c_p} z}{T_s}\right)$$

$$\Rightarrow \boxed{\frac{p}{p_s} = \left(\frac{T}{T_s}\right)^{\frac{M_a c_p}{R}}}$$

Then $\rho = \frac{M_a p}{RT} = \frac{M_a p_s}{R T_s}\left(\frac{p}{p_s}\right)\left(\frac{T_s}{T}\right) = \frac{M_a p_s}{R T_s}\left(\frac{T}{T_s}\right)^{\frac{M_a c_p}{R} - 1}$ (note $\frac{M_a c_p}{R} \approx 3.5$)

$$\Rightarrow \boxed{\rho = \underbrace{\frac{M_a p_s}{R T_s}}_{\rho_s} \left(\frac{T_s - \frac{g}{c_p} z}{T_s}\right)^{\frac{M_a c_p}{R} - 1}}$$

Since density, temp, and pressure all tend to zero at $z = d = \frac{c_p T_s}{g}$, this is a natural definition for the top of the atmosphere. If $T_s = 300$ K, this is around 30 km.

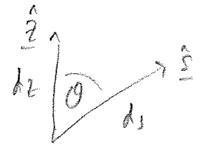


$\frac{\rho}{\rho_s} = \left(\frac{T}{T_s}\right)^{\frac{M_a c_p}{R} - 1}$ and $T = T_s - \frac{g}{c_p} z$. For Mt Everest, $z = 8{,}848$ m and $T_s \approx 300$ K, so $\frac{T}{T_s} \approx \frac{2}{3}$ Hence $\frac{\rho}{\rho_s} \approx \left(\frac{2}{3}\right)^{2.5} \approx 0.36$, so around <u>30 - 40%</u> thinner

Note: This question ignores the stratosphere, in which absorption of shortwave radiation by ozone causes the temperature to rise again. The behaviour at large heights, including the prediction for the depth of the atmosphere, is therefore not correct. It is nevertheless interesting to see what the model predicts.

2. Two stream approximation

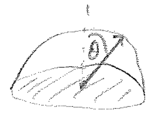
(i) $\cos\theta \frac{\partial I}{\partial z} = -\kappa\rho(I-B)$ $B=\frac{\sigma T^4}{\pi}$



[$dz=\cos\theta\, ds$.
$\frac{\partial I}{\partial s} = -\kappa\rho(I-B)$ for $I(x,\hat s)$]

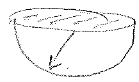
Write $\mu=\cos\theta$, so $\boxed{\mu\frac{\partial I}{\partial z}=-\kappa\rho(I-B)}$ (RTE).

Define $I_+ = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\pi/2} I \underbrace{\sin\theta\,d\theta}_{-d\mu}\,d\phi$ i.e. average over upward-pointing directions



$= \int_0^1 I\,d\mu$

and similarly $I_- = \frac{1}{2\pi}\int_0^{2\pi}\int_{\pi/2}^{\pi} I\sin\theta\,d\theta\,d\phi = \int_{-1}^0 I\,d\mu$.



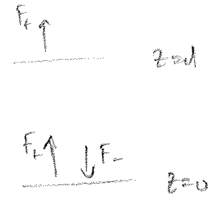
The total upward and downward fluxes are given by

$$F_+ = \int_0^{2\pi}\int_0^{\pi/2} I\cos\theta\sin\theta\,d\theta\,d\phi = 2\pi\int_0^1\mu I\,d\mu \approx \pi I_+$$

$$F_- = -\int_0^{2\pi}\int_{\pi/2}^{\pi} I\cos\theta\sin\theta\,d\theta\,d\phi = -2\pi\int_{-1}^0 \mu I\,d\mu \approx \pi I_-$$

Then integrating RTE×π ($\int_0^1 d\mu$ and $\int_{-1}^0 d\mu$) gives

$$\boxed{\frac{1}{2}\frac{dF_+}{dz}\approx -\kappa\rho(F_+-\pi B) \qquad -\frac{1}{2}\frac{dF_-}{dz}\approx-\kappa\rho(F_--\pi B)}$$



At $z=d$ (top of atmosphere) there is no incoming radiation, so $\boxed{F_-=0 \text{ at } z=d.}$

At $z=0$ (surface), Stefan-Boltzmann law gives $\boxed{F_+=\sigma T_s^4 \text{ at } z=0.}$ Also $\boxed{F_+=\sigma T_e^4 \text{ at } z=d.}$

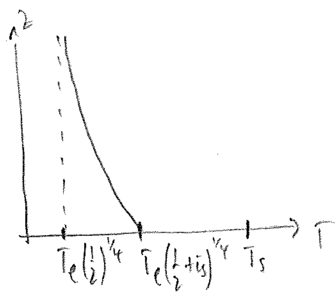
(ii) Local radiative equilibrium means $B=\frac{1}{2}(I_++I_-) \Rightarrow \pi B=\frac{1}{2}(F_++F_-)$, so $\boxed{\frac{dF_+}{dz}=-\kappa\rho(F_+-F_-)=\frac{dF_-}{dz}}$

Hence $\frac{d}{dz}(F_+-F_-)=0$, so $F_+-F_-=F$ is constant ($=\sigma T_e^4$ from $z=d$)

Then $\frac{dF_-}{dz}=-\kappa\rho F \Rightarrow \boxed{F_-=F\int_z^d\kappa\rho\,dz = F\tau.}$ (using $F_-=0$ at $z=d$).

and $\boxed{F_+=F(1+\tau)}$ [$\tau=\int_z^d \kappa\rho\,dz$ optical depth]

So $\pi B=\sigma T^4 = F(\frac{1}{2}+\tau)=\sigma T_e^4(\frac{1}{2}+\tau) \Rightarrow \boxed{T=T_e\left(\frac{1}{2}+\tau\right)^{1/4}}$



At $z=0$, $F_+=\sigma T_s^4 = F(1+\tau_s)=\sigma T_e^4(1+\tau_s) \Rightarrow \boxed{T_e=T_s(1+\tau_s)^{-1/4}}$

$\tau_s=\int_0^d\kappa\rho\,dz$

i.e. $\boxed{\gamma=\frac{1}{(1+\tau_s)^{1/4}}}$

(iii) If $T(z)$ is known, take $\frac{1}{2}\frac{dF_\uparrow}{dz} = -\kappa\rho\,(F_\uparrow - \sigma T^4)$

Let $\tau = \int_z^d \kappa\rho\, dz$ so $\frac{d}{dz} = -\kappa\rho\frac{d}{d\tau}$. Then $\boxed{\frac{dF_\uparrow}{d\tau} - 2F_\uparrow = -2\sigma T^4}$

Integrating factor $e^{-2\tau}$ $\Rightarrow$ $\frac{d}{d\tau}\left(F_\uparrow e^{-2\tau}\right) = -\sigma T^4 e^{-2\tau}$

$\Rightarrow$ $F_\uparrow e^{-2\tau} - \sigma T_s^4 e^{-2\tau_s} = \int_\tau^{\tau_s} 2\sigma T^4 e^{-2\hat\tau}\, d\hat\tau$ (using $F_\uparrow = \sigma T_s^4$ at $\tau = \tau_s$)

$\Rightarrow$ $\boxed{F_\uparrow = \sigma T_s^4\left[e^{-2(\tau_s-\tau)} + \int_\tau^{\tau_s} 2\left(\frac{T}{T_s}\right)^4 e^{-2(\hat\tau-\tau)}\, d\hat\tau\right]}$

At $\tau = 0$ $(z=d)$, $F_\uparrow = \sigma T_e^4$, so $T_e^4 = T_s^4\left[e^{-2\tau_s} + \int_0^{\tau_s} 2\left(\frac{T}{T_s}\right)^4 e^{-2\hat\tau}\, d\hat\tau\right]$.

i.e. $\boxed{\gamma = e^{-2\tau_s} + \int_0^{\tau_s} 2\left(\frac{T}{T_s}\right)^4 e^{-2\tau}\, d\tau}$

(iv) From Q1, $\left(\frac{T}{T_s}\right) = \left(\frac{p}{p_s}\right)^{\frac{R}{M_a c_p}}$ and $p = \int_z^d \rho g\, dz$.

Note that $\tau = \int_z^d \kappa\rho\, dz$, so if κ is constant, then $\boxed{\tau = \frac{\kappa}{g}p}$ and hence $\frac{p}{p_s} = \frac{\tau}{\tau_s}$.

So $\boxed{\left(\frac{T}{T_s}\right) = \left(\frac{\tau}{\tau_s}\right)^{\frac{R}{M_a c_p}}}$

Hence $\boxed{\gamma = e^{-2\tau_s} + \int_0^{\tau_s} 2\left(\frac{\tau}{\tau_s}\right)^{\frac{4R}{M_a c_p}} e^{-2\tau}\, d\tau.}$

For small τ_s, $\gamma = e^{-2\tau_s} + 2\tau_s\int_0^1 x^{\frac{4R}{M_a c_p}} e^{-2\tau_s x}\, dx$. $[\tau = \tau_s x]$.

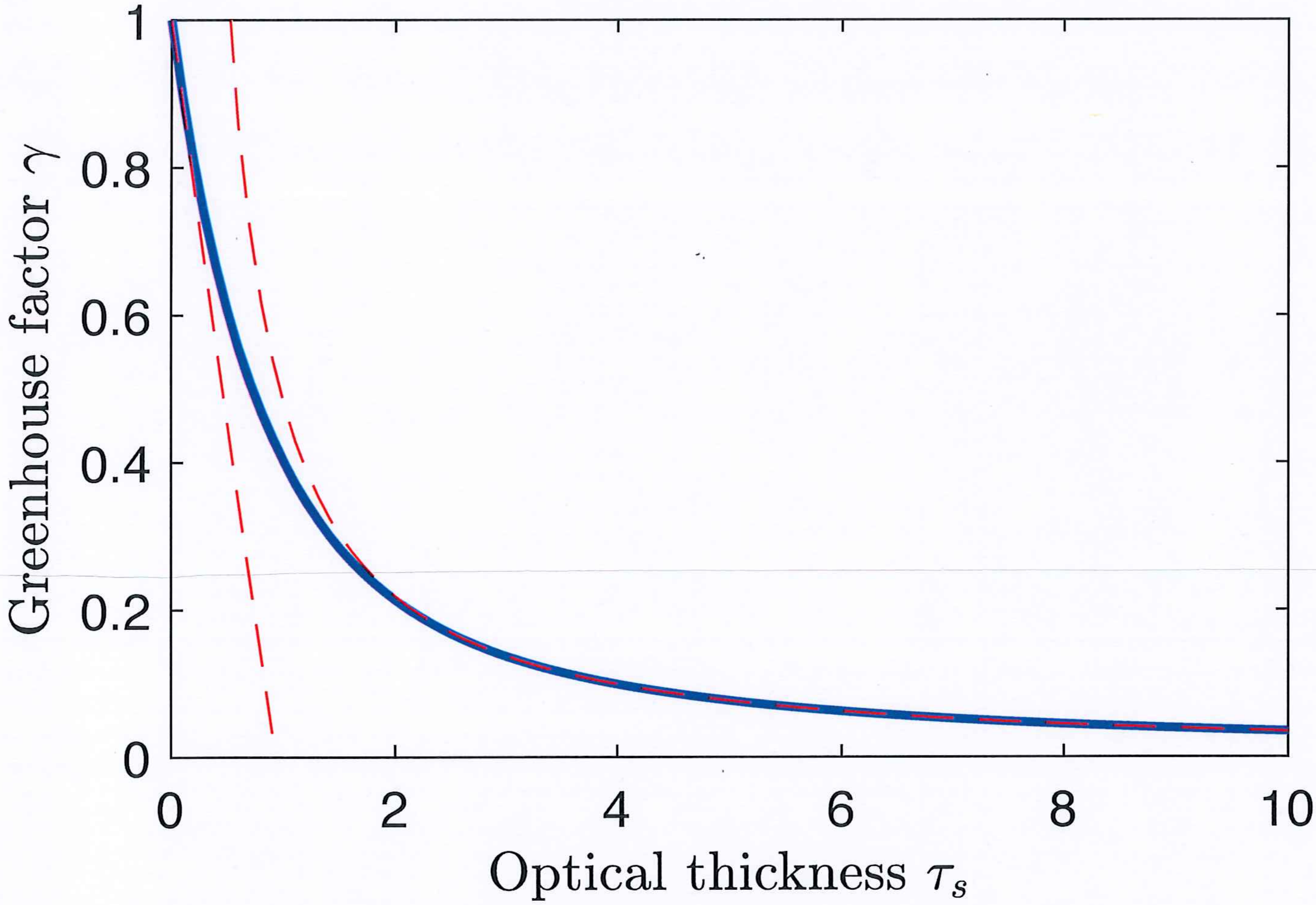
$\sim 1 - 2\tau_s + \dots + 2\tau_s\left(\int_0^1 x^{\frac{4R}{M_a c_p}}\, dx + O(\tau_s)\right)$ $\quad \int_0^1 x^\alpha\, dx = \frac{1}{\alpha+1}$

$= 1 - 2\tau_s + 2\tau_s\frac{M_a c_p}{4R + M_a c_p} + O(\tau_s^2) = \boxed{1 - \frac{8R}{4R + M_a c_p}\tau_s + O(\tau_s^2)}$

For large τ_s, first term is exponentially small, and limit in integral can be replaced by ∞ with exponentially small errors, so $\gamma \sim \int_0^\infty \left(\frac{t}{2\tau_s}\right)^{\frac{4R}{M_a c_p}} e^{-t}\, dt$ $[2\tau = t]$

$= \boxed{(2\tau_s)^{-\frac{4R}{M_a c_p}} \underbrace{\int_0^\infty t^{\frac{4R}{M_a c_p}} e^{-t}\, dt}_{\Gamma\left(1 + \frac{4R}{M_a c_p}\right)}}$

[See greenhouse_factor.m for numerical calculation of γ]


Greenhouse factor γ
Optical thickness τ_s
0
0.2
0.4
0.6
0.8
1
0
2
4
6
8
10

3. Runaway greenhouse effect.

(i) Clausius Clapeyron equation $\frac{\partial p_{sv}}{\partial T} = \frac{\rho_v L}{T}$ $\quad p_{sv} = p_{sv0}$ at $T = T_0$.

$$= \frac{p_{sv}}{T^2}\frac{M_v L}{R} \qquad \left[\text{using } p_{sv} = \frac{\rho_v R T}{M_v}\right]$$

[integrate] $\Rightarrow$ $$\frac{dp_{sv}}{p_{sv}} = \frac{M_v L}{R}\frac{dT}{T^2}$$

$$\ln\frac{p_{sv}}{p_{sv0}} = \frac{M_v L}{R}\left\{\frac{1}{T_0} - \frac{1}{T}\right\}$$

$$\Rightarrow \boxed{p_{sv} = p_{sv0}\, e^{a\left(1-\frac{T_0}{T}\right)}} \qquad \boxed{a = \frac{M_v L}{R T_0}}$$

If $T - T_0 \ll T_0$ write $1 - \frac{T_0}{T} = 1 - \frac{1}{1+\frac{T-T_0}{T_0}} = 1 - \left(1 - \frac{T-T_0}{T_0} + \dots\right) = \frac{T-T_0}{T_0} + O\left(\frac{T-T_0}{T_0}\right)^2$

(ii) Energy balance $\frac{1}{4}Q = \sigma\gamma T^4$ $\Rightarrow$ $\boxed{T = \left(\frac{Q}{4\sigma\gamma}\right)^{\frac{1}{4}} = \left(\frac{Q}{4\sigma}\right)^{\frac{1}{4}}\left(1 + b\left(\frac{p_v}{p_{sv0}}\right)^c\right)}$

Write $\boxed{\theta = \frac{T}{T_0}}$, then energy balance gives $\theta = \underbrace{\left(\frac{Q}{4\sigma T_0^4}\right)^{\frac{1}{4}}}_{\alpha}\left(1 + b\left(\frac{p_v}{p_{sv0}}\right)^c\right)$

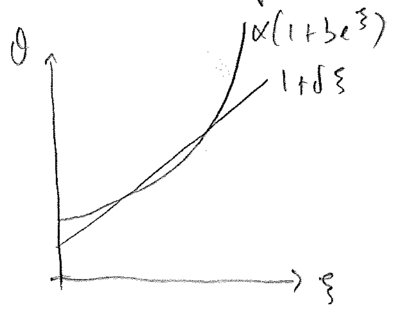
$$\Rightarrow \boxed{\theta = \alpha(1 + b e^{\xi_v})} \quad \text{defining} \quad \boxed{\xi_v = c\ln\frac{p_v}{p_{sv0}}}$$

With the same notation, the saturation curve becomes. $\frac{\xi_{sv}}{c} = a(\theta - 1)$

(but $\xi_{sv} = c\ln\frac{p_{sv}}{p_{sv0}}$)

$$\Rightarrow \boxed{\theta = 1 + \delta\xi_{sv}} \qquad \delta = \frac{1}{ac}.$$

(iii) The runaway greenhouse effect occurs if the temperature calculated from energy balance remains above the saturation curve for all p_v (ie ξ_v). So its occurrence depends on the non-intersection of $\theta = \alpha(1 + be^{\xi})$ with $\theta = 1 + \delta\xi$.



Clearly from the graph, non intersection will happen if α is large enough.
The critical α is found from when the curves meet tangentially, ie. $\boxed{\begin{aligned}\alpha(1+be^{\xi}) &= 1 + \delta\xi \\ \& \quad \alpha b e^{\xi} &= \delta.\end{aligned}}$

$$\Rightarrow \boxed{\alpha + \delta = 1 + \delta\xi = 1 + \delta\ln(\delta/\alpha b)}$$

If δ is small then, $\alpha \hat{=} 1$, and putting this back into the right hand side gives improved estimate

$$\alpha \approx 1 + \delta\ln(\delta/b) - \delta \qquad [\text{The corrections are } O(\delta^2 \ln\delta)\ ]$$

(iv) For the Earth $Q = 1370\ \mathrm{W m^{-2}}$, $\sigma = 5.67 \times 10^{-8}\ \mathrm{W m^{-2} K^{-1}}$, $T_0 = 273\ \mathrm{K}$.

So $\boxed{\alpha = \left(\frac{Q}{4\sigma T_0^4}\right)^{\frac{1}{4}} \approx 1.02119.}$

$R = 8.3\ \mathrm{J mol^{-1} K^{-1}}$, $M_v = 18 \times 10^{-3}\ \mathrm{kg mol^{-1}}$, $L = 2.5 \times 10^6\ \mathrm{J kg^{-1}}$ $\Rightarrow$ $a \approx 19.9$.

$\Rightarrow$ $\delta \approx 0.2$ (using $c = 0.25$).

So $\boxed{\alpha_c \approx 1.05.}$

So $\alpha < \alpha_c$, suggesting runaway greenhouse effect does not occur.

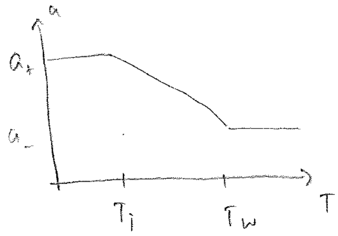
For Venus, Q is twice as large so α is increased by a factor of $2^{1/4}$ $\Rightarrow$ $\boxed{\alpha \approx 1.21}$

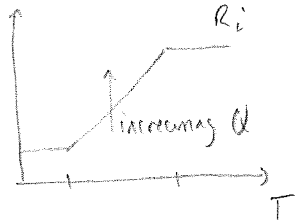
This is larger than α_c, so runaway greenhouse effect does occur. (ie the saturation vapour pressure is never reached).

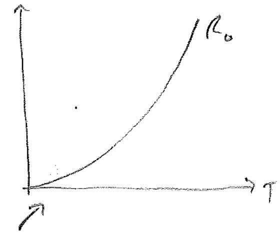
x If solar radiation were 30% smaller when the atmospheres were forming, this would not make a difference, since decreasing α by $(0.7)^{1/4} \approx 0.91$ does not change the conclusion that $\alpha < \alpha_c$ for Earth and $\alpha > \alpha_c$ for Venus.

4. <u>Ice-albedo feedback.</u> $c\frac{dT}{dt} = R_i - R_o$. $R_i = \frac{1}{4}Q(1-a)$ $R_o = \sigma\gamma T^4$

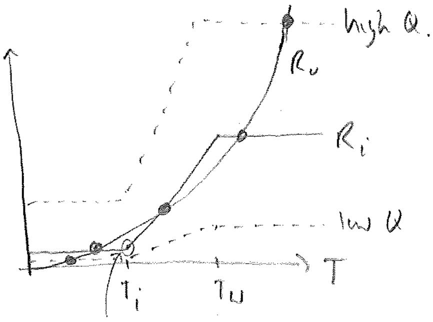
(i)







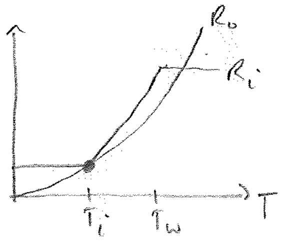
Steady states are intersection of these

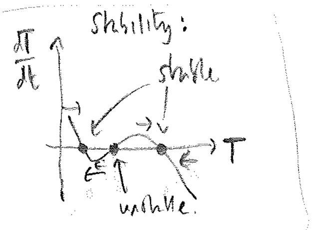


From the graph it is clear that there can be multiple intersections for intermediate values of Q, provided the slope of the central section of the R_i curve is sufficiently steep.

In particular, multiple intersections require $R_i(T_i) < R_o(T_i)$, and the slope $R_i'(T_i) = \frac{1}{4}Q\frac{a_+-a_-}{T_w-T_i}$ must be larger than $R_o'(T_i) = 4\sigma\gamma T_i^3$. The largest value that $R_i'(T_i)$ takes, while this point remains below the $R_o(T_i)$ curve is when $R_i(T_i) = R_o(T_i) \Rightarrow \frac{1}{4}Q(1-a_+) = \sigma\gamma T_i^4$

$\Rightarrow \Big| Q = \frac{4\sigma\gamma T_i^4}{1-a_+}$, so we require $\frac{\sigma\gamma T_i^4}{1-a_+}\frac{a_+-a_-}{T_w-T_i} > 4\sigma\gamma T_i^3$

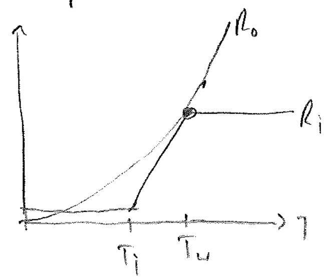




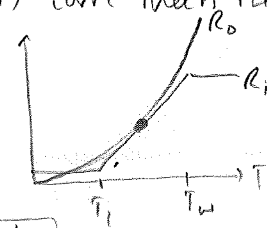
$$\Leftrightarrow \boxed{\frac{T_w-T_i}{T_i} < \frac{a_+-a_-}{4(1-a_+)}}$$

(equality would occur when the two slopes are equal)

(ii) If this condition on the slopes held, then for smaller Q than this value $\left(\boxed{Q_+ = \frac{4\sigma\gamma T_i^4}{1-a_+}}\right)$ there will clearly be multiple intersections (as in diagram above). As Q is reduced, the multiple intersections cease to occur either when $R_i(T_w)$ drops below $R_o(T_w)$:



or when the $R_i(T)$ curve meets the $R_o(T)$ curve tangentially:



In the first case, the lower bound is $\boxed{Q_- = \frac{4\sigma\gamma T_w^4}{1-a_-}}$ and this applies if $R_i'(T_w) > R_o'(T_w)$ then

i.e. $\frac{\sigma\gamma T_w^4}{1-a_-}\frac{a_+-a_-}{T_w-T_i} > 4\sigma\gamma T_w^3$

$$\Leftrightarrow \boxed{\frac{T_w-T_i}{T_w} < \frac{a_+-a_-}{4(1-a_-)}}$$

In the second case, we must find the value of Q for which the curves meet tangentially.

This happens when
$$\left.\begin{aligned} \tfrac{1}{4}Q(1-a) &= \sigma\gamma T^4 \\ \tfrac{1}{4}Q\,\frac{a_+-a_-}{T_w-T_i} &= 4\sigma\gamma T^3 \end{aligned}\right\}$$ Solve for T and Q.
$$1-a = 1-a_+ + \frac{a_+-a_-}{T_w-T_i}(T-T_i)$$

Write $\lambda = \dfrac{a_+-a_-}{T_w-T_i}$, then
$$\left.\begin{aligned} \tfrac{1}{4}Q(1-a_+ + \lambda(T-T_i)) &= \sigma\gamma T^4 \\ \tfrac{1}{4}Q\lambda &= 4\sigma\gamma T^3 \end{aligned}\right\}$$
$$1-a_+ + \lambda(T-T_i) = \frac{\lambda T}{4}.$$
$$\Rightarrow \frac{3}{4}T = T_i - \frac{(1-a_+)}{\lambda}.$$
$$\Rightarrow T = \frac{4}{3}T_i - \frac{4}{3}\frac{(1-a_+)}{\lambda} = \frac{4}{3}\left[\frac{(1-a_-)T_i - (1-a_+)T_w}{a_+-a_-}\right]$$

Then
$$Q = \frac{16\sigma\gamma T^3}{\lambda} = \frac{16\sigma\gamma T^3(T_w-T_i)}{a_+-a_-} = \frac{64\cdot 16\sigma\gamma}{27}\frac{(T_w-T_i)}{(a_+-a_-)^4}\left[(1-a_-)T_i - (1-a_+)T_w\right]^3.$$
$$\Rightarrow \boxed{Q_- = \frac{\cancel{512}\ 1024}{27}\sigma\gamma\,\frac{(T_w-T_i)\left[(1-a_-)T_i - (1-a_+)T_w\right]^3}{(a_+-a_-)^4}}$$

(This can be seen to give the same value as in the first case if $\dfrac{T_w-T_i}{T_w} = \dfrac{a_+-a_-}{4(1-a_-)}$)