Sheet 0

1. Steady state has $\frac{1}{4}Q(1-a) = \sigma T^4 \Rightarrow T = \left[\frac{Q(1-a)}{4\sigma}\right]^{1/4}$.

$Q = \frac{Q_{Earth}}{R^2}$ where R is distance from sun to planet in astronomical units (since energy flux from sun falls off according to inverse square law;

So plugging in numbers, $T_{Earth} \approx 255$ K
$T_{Venus} \approx 228$ K
$T_{Mars} \approx 217$ K
$T_{Jupiter} \approx 98$ K.

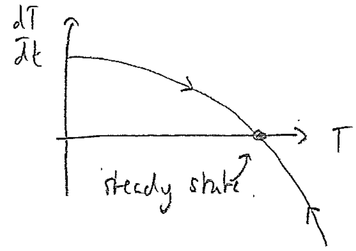
Discrepancies are likely to be due to the greenhouse effect. and perhaps (in the case of Jupiter) to internal heat generation.

The stability of the steady state $\bar{T}$ can be examined by writing $T = \bar{T} + \theta$ where $\theta \ll \bar{T}$,

then $\rho c d \frac{d\theta}{dt} \approx -4\sigma \bar{T}^3 \theta$ (linearising)

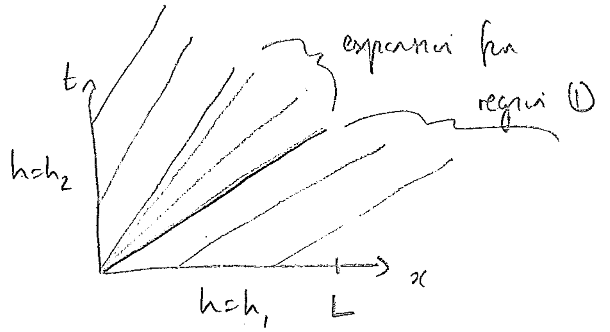
$\Rightarrow$ stable (since any initial perturbation θ' will decay to zero)

Alternatively, plot $\frac{dT}{dt}$ as function of T



Trajectories approach steady state

2. $\frac{\partial h}{\partial t} + h^m \frac{\partial h}{\partial x} = 0$



Characteristic equations $\dot{t} = 1$ $\quad \dot{x} = h^m$ $\quad \dot{h} = 0$ $\qquad \dot{} =$ derivative along characteristic

Initial conditions for characteristics from $t=0$: $\quad t=0,\ x = x_0,\ h = h_1$

$$\Rightarrow \boxed{h=h_1},\quad x = x_0 + h_1^m t$$

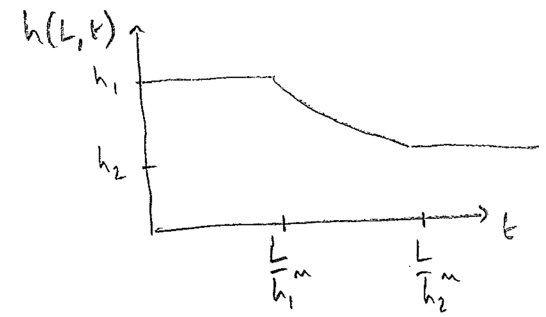
so these characteristics sweep out region (1) on diagram.

'Initial' conditions for characteristics from $x=0$: $\quad t = t_0,\ x=0,\ h = h_2$

$$\Rightarrow \boxed{h=h_2},\quad x = h_2^m (t - t_0)$$

sweep out region (2) on diagram.

In between regions (1) and (2) there must be an expansion fan, where characteristics come from the origin. From the characteristic equations h is constant along each characteristic, which therefore have equations $x = h^m t$. So $\boxed{h = \left(\frac{x}{t}\right)^{\frac{1}{m}}}$



If $h_2 > h_1$, a shock will form (because the two sets of characteristics intersect)

This will move with speed given by Rankine-Hugoniot conditions

$$[h]_-^+ V = \left[\frac{1}{m+1} h^{m+1}\right]_-^+ \quad \text{ie} \quad V = \frac{h_2^{m+1} - h_1^{m+1}}{(m+1)(h_2 - h_1)}$$

3. $\rho c \dfrac{\partial T}{\partial t} = k \dfrac{\partial^2 T}{\partial z^2}$ $\qquad \dfrac{\partial T}{\partial z} \to 0$ as $z \to \infty$. $\qquad T = T_0 - \Delta T \cos \omega t$.

Scale $\quad T = T_0 + \Delta T \, \hat{T}$ $\qquad$ ^ variables are dimensionless.

$\quad z = [z] \hat{z}$

$\quad t = [t] \hat{t}$ $\qquad$ Choose $[t] = \dfrac{1}{\omega}$. $\quad [z] = \left(\dfrac{k}{\rho c \omega} \right)^{1/2}$.

(to achieve balance of terms indicated by arrows).

$$\frac{\rho c [T]}{[t]} \frac{\partial \hat{T}}{\partial \hat{t}} = \frac{k [T]}{[z]^2} \frac{\partial^2 \hat{T}}{\partial \hat{z}^2}$$

$\Rightarrow \quad \dfrac{\partial T}{\partial t} = \dfrac{\partial^2 T}{\partial z^2}$ with $\dfrac{\partial T}{\partial z} \to 0$ as $z \to \infty$ $\quad T = -\cos t$. $\qquad$ (dropped hats).

$T = -\mathrm{Re}\left(e^{it}\right)$ at $z=0$, so look for solution of form $T = -\mathrm{Re}\left(e^{it} f(z)\right)$

$\Rightarrow \quad -if = f''$ with $f' \to 0$ as $z \to \infty$, $f = 1$ at $z = 0$.

$\Rightarrow \quad f = e^{-\frac{1+i}{\sqrt{2}} z} = e^{-z/\sqrt{2}} \, e^{-iz/\sqrt{2}}$.

so $\quad \boxed{T = -e^{-z/\sqrt{2}} \cos\left(t - z/\sqrt{2}\right).}$

Temperature decays over dimensionless distance $\sim \sqrt{2}$, so dimensional distance is $z \sim \left(\dfrac{2k}{\rho c \omega} \right)^{1/2} \sim \left(\dfrac{kP}{\rho c \pi} \right)^{1/2}$ where $P = \dfrac{2\pi}{\omega}$ is the period of the forcing.

For $\rho = 1600\ \mathrm{kg\,m^{-3}}$, $c = 800\ \mathrm{J\,kg^{-1}K^{-1}}$, $k = 1\ \mathrm{W\,m^{-1}K^{-1}}$,

$\quad P = 1\,\mathrm{d} \to z \sim 0.15\ \mathrm{m}$

$\quad P = 1\,\mathrm{y} \to z \sim 2.8\ \mathrm{m}$

For ice ($\rho = 900\ \mathrm{kg\,m^{-3}}$, $c = 2000\ \mathrm{J\,kg^{-1}K^{-1}}$, $k = 2\ \mathrm{W\,m^{-1}K^{-1}}$) the numbers change by only a small amount (a factor of $\left(\dfrac{K_{ice}}{K_{soil}} \right)^{1/2} \approx 1.2$) to $z \sim 0.17\ \mathrm{m}$, $z \sim 3.3\ \mathrm{m}$.

This would not be a good model for the temperature near the surface of a lake because it is likely that the temperature variables cause convection (due to density changing with temperature), which the heat equation ignores.

4. $$B_\nu = \frac{2h\nu^3}{c^2\left(e^{h\nu/kT}-1\right)} \qquad u = \frac{h\nu}{kT} \qquad du = \frac{k}{kT}\,d\nu.$$

$$B = \int_0^\infty B_\nu\,d\nu = \int_0^\infty \frac{2h}{c^2}\left(\frac{kT}{h}\right)^3 \frac{u^3}{e^u-1}\left(\frac{kT}{h}\right)du$$

$$= \frac{2k^4T^4}{c^2h^3}\underbrace{\int_0^\infty \frac{u^3}{e^u-1}\,du}_{I}$$

$$I = \int_0^\infty \frac{u^3e^{-u}}{1-e^{-u}}\,du = \int_0^\infty \sum_{n=0}^\infty u^3e^{-u}e^{-nu}\,du$$ (Binomial expansion on $\frac{1}{1-e^{-u}}$) converges uniformly.

$$= \sum_{n=1}^\infty \int_0^\infty u^3e^{-nu}\,du \qquad nu = v \qquad du = \frac{1}{n}dv$$

$$= \sum_{n=1}^\infty \frac{1}{n^4}\underbrace{\int_0^\infty v^3e^{-v}\,dv}_{=3!=6.}$$

$$= 6\sum_{n=1}^\infty \frac{1}{n^4}$$

$$= \frac{\pi^4}{15}.$$

Hence $B = \frac{2k^4T^4}{c^2h^3}\frac{\pi^4}{15} = \frac{\sigma T^4}{\pi}$ where $\sigma = \frac{2\pi^5k^4}{15h^3c^2}$