C5.7 [C6.4a] 1

Topics in fluids 2008 q2 answer (cf notes)

This seems to be a rather poorly written question.

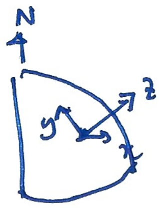
$$\frac{d\rho}{dt} + \rho \underline{\nabla} \cdot \underline{u} = 0 \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + \underline{u} \cdot \underline{\nabla}$$

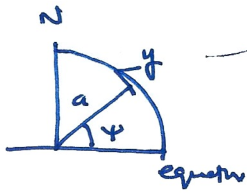
$$\frac{du}{dt} + fv = -\frac{1}{\rho} p_x$$

$$\frac{dv}{dt} - fu = -\frac{1}{\rho} p_y$$

$$g = -\frac{1}{\rho} p_z$$

$$\frac{d\theta}{dt} = 0$$





These are dimensional equations

The perfect gas law is $p = \frac{\rho R T}{M}$ $\quad$ M = molecular weight

Dr says $\theta = \frac{T}{p^{\nu}}$ – this is not dimensionless !!

Revert to dimensional terms: dry adiabat is $\rho c_p \frac{dT}{dt} - \alpha T \frac{dp}{dt} = 0$

& $\alpha = \frac{1}{T}$ so $\rho c_p \, dT = dp$

& $\rho = \frac{Mp}{RT}$ $\Rightarrow$ $\frac{M c_p}{RT} p \, dT = dp$ $\Rightarrow$ $\ln T = \frac{R}{M c_p} \ln p$

or $\frac{T}{p^{\nu}}$ = constant $\quad \nu = \frac{R}{M c_p}$ $\quad$ & define potential temperature

$$\theta = T\left(\frac{p_0}{p}\right)^{\nu}$$

we can take p_0 = pressure at Earth's surface

(i) $\underline{u} = 0$ & assume also $\frac{\partial}{\partial t} = 0 \Rightarrow \frac{d}{dt} = 0$

$\Rightarrow p = p(z)$ & assume $\theta = T_0$ $\quad \left(\frac{\theta}{T_0} = 1\right)$ $\quad T_0$ = surface temperature

So $$\rho = \frac{Mp}{RT} = \frac{Mp}{RT_0}\left(\frac{p_0}{p}\right)^{\nu} = \frac{M p_0^{\nu}}{RT_0} \cancel{p} \frac{1}{p^{\nu-1}}$$

$$p_z = -\rho g = -\frac{Mg}{RT_0}\frac{p_0^{\nu}}{p^{\nu-1}}$$

$$\Rightarrow \frac{p^{\nu}}{\nu} = \frac{p_0^{\nu}}{\nu} - \frac{Mg}{RT_0} p_0^{\nu} z \qquad \text{if } p = p_0 \text{ at } z = 0$$

$$\left(\frac{p}{p_0}\right)^{\nu} = 1 - \frac{Mg}{R} \cdot \frac{R}{Mc_pT_0} z$$

thus $$\underline{\frac{p}{p_0} = \left[1 - \frac{z}{h}\right]^{\frac{1}{\nu}}}, \quad \nu = \frac{R}{Mc_p}, \quad h = \frac{c_pT_0}{g}$$

eg
$R = 8.3$ J mole^{-1} K^{-1}
$M = 28.8$ ~~kg~~ $\times 10^{-3}$ kg mole^{-1}
$c_p = 10^3$ J kg^{-1} K^{-1}
$T_0 \simeq 290$ K
$g \simeq 10$ m s^{-2}

$$\Rightarrow \nu \sim \frac{8.3}{28.8 \; 10^{-3} 10^{3}} \frac{\text{J mole}^{-1}\text{K}^{-1} \text{ kg K}}{\text{kg mole}^{-1} \text{ J}} \simeq 0.3$$

$$h \simeq \frac{10^3 \, 290}{10} \frac{\text{J kg}^{-1}\text{K}^{-1}\text{K}}{\text{m s}^{-2}} \sim 29 \text{ km}$$

[or $$\frac{p}{p_0} = \left[1 - \frac{\nu z}{H}\right]^{\frac{1}{\nu}} \approx \exp\left(-\frac{z}{H}\right) \text{ for small } \nu$$

& $$H = \nu h = \frac{R}{Mc_p}\frac{c_pT_0}{g} = \frac{RT_0}{Mg} \sim 8.4 \text{ km}$$ — scale height

~~8.3 290 J mole^{-1} K^{-1} K~~

(ii) I think what is required here is the static [Pedlosky 1st ed, §6.4] stability argument: If a parcel of air at z is moved rapidly (adiabatically) to $z+\Delta z$ its density will change adiabatically

to $\rho = \bar{\rho}(z) + \Delta\rho_A$ ($\bar{\rho}(z)$ is basic density profile)

It is convenient & sensible to define $p_0 = \frac{\rho_0 R T_0}{M}$ (ρ_0 is surface density)

$\rightarrow \rho_0 = \frac{M p_0}{RT}$

so $$\frac{\rho}{\rho_0} = \frac{1}{\rho_0}\frac{M p_0}{RT}\left(\frac{p_0}{p}\right)^{\nu-1} = \left(\frac{p}{p_0}\right)^{1-\nu}$$

So $$\frac{\Delta\rho_A}{\rho_0} = \frac{(1-\nu)}{p}\left(\frac{p}{p_0}\right)^{1-\nu}\Delta p = \frac{(1-\nu)}{p}\frac{\rho}{\rho_0}\Delta p$$
$$= -\frac{(1-\nu)}{p}\frac{\rho}{\rho_0}\rho g\,\Delta z$$

on the other hand the density of the ambient air at $z+\Delta z$

is $$\bar{\rho}(z+\Delta z) = \bar{\rho}(z) + \frac{\partial\bar{\rho}}{\partial z}\Delta z$$

Therefore the buoyancy force upwards per unit volume is
$$g\left[\frac{\partial\bar{\rho}}{\partial z} + \frac{(1-\nu)}{p}\frac{\rho}{\rho_0}\rho g\right]\Delta z$$

Now $$\bar{\rho} = \rho_0\left(\frac{p}{p_0}\right)^{\cancel{*}1-\nu} = \rho_0\left[1-\frac{z}{h}\right]^{\frac{1-\nu}{\nu}} \qquad \left(\frac{p}{p_0} = \left[1-\frac{z}{h}\right]^{\frac{1}{\nu}}\right)$$

So $$\frac{\partial\bar{\rho}}{\partial z} = \frac{1-\nu}{\nu}\rho_0\left[1-\frac{z}{h}\right]^{\frac{1-\nu}{\nu}-1}\cdot -\frac{1}{h}$$

& note $h = \frac{c_p T_0}{g} \varepsilon_0$

$$\frac{1}{h} = \frac{g}{c_p T_0} = \frac{g\cancel{R}}{\cancel{Mc_p}} \; g\left(\frac{R}{Mc_p}\right)\frac{M}{RT_0}$$

$$= \nu g \frac{\rho_0 M}{\rho_0 R T_0} = \frac{\nu g \rho_0}{p_0}$$

So
$$\frac{\partial \bar{p}}{\partial z} = -\frac{(1-\nu)}{\nu}\left[1-\frac{z}{h}\right]^{\frac{1-\nu}{\nu}-1} \frac{\nu g \rho_0}{p_0}$$

$$= -(1-\nu) g \frac{\rho_0}{p_0}\left[1-\frac{z}{h}\right]^{\frac{1-\nu}{\nu}-1}$$

So the net upwards force per unit volume is

$$g\left[-(1-\nu) g \frac{\rho_0}{p_0}\left\{1-\frac{z}{h}\right\}^{\frac{1-\nu}{\nu}-1} + \frac{(1-\nu)}{p}\frac{\rho}{\rho_0}\rho g\right]\Delta z$$

$$= \cancel{g}\,(1-\nu) g^2\left[-\frac{\rho_0}{p_0}\left\{1-\frac{z}{h}\right\}^{\frac{1-\nu}{\nu}-1} + \frac{1}{\rho_0}\rho_0^2\left\{1-\frac{z}{h}\right\}^{\frac{2(1-\nu)}{\nu}}\frac{1}{p_0\left\{1-\frac{z}{h}\right\}^{\frac{1}{\nu}}}\right]$$

$\frac{1}{p}$

$$= \frac{\rho_0}{p_0}(1-\nu) g^2\left[-\left\{1-\frac{z}{h}\right\}^{\frac{1}{\nu}-2} + \left\{1-\frac{z}{h}\right\}^{\frac{1}{\nu}-2}\right]$$

ρ^2

$$= 0$$

$\Rightarrow$ neutral stability

bit of a mouthful & could probably be better done iii This is heading towards Ertel's theorem.

Note we have $\theta = T\left(\frac{p_0}{p}\right)^{\nu}$, $\nu = \frac{R}{Mc_p}$

$$\frac{\theta}{T_0} = \frac{T}{T_0}\left(\frac{p_0}{p}\right)^{\nu}, \qquad \rho = \frac{Mp}{RT} \Rightarrow \frac{\rho}{\rho_0} = \frac{p_0 M}{\rho_0 R T}\,\frac{p}{p_0}$$

$$= \frac{T_0}{T}\,\frac{p}{p_0}$$

$$\frac{T}{T_0} = \frac{p}{p_0} \Big/ \frac{\rho}{\rho_0} \Leftarrow \quad \text{as } p_0 = \rho_0 R T_0$$

So $\frac{\theta}{T_0} = \frac{(p/p_0)^{1-\nu}}{(\rho/\rho_0)}$ in particular $\theta = \theta(p, \rho)$

So $\underline{\nabla}\theta = \theta_p \underline{\nabla} p + \theta_\rho \underline{\nabla}\rho$

I think there is an enormous typo here. I think what is meant is, show $\underline{\nabla}\theta \cdot \underline{\nabla}\times\left[\frac{1}{\rho}\underline{\nabla} p\right] = 0$

In which case since $\text{curl}\, \phi \underline{a} = \phi\, \text{curl}\, \underline{a} + \underline{\nabla}\phi \times \underline{a}$

$$\text{curl}\, \frac{1}{\rho}\underline{\nabla} p = \underline{\nabla}\left(\frac{1}{\rho}\right)\times\underline{\nabla} p = -\frac{1}{\rho^2}\underline{\nabla}\rho\times\underline{\nabla} p$$

and then $\underline{\nabla}\theta \cdot \underline{\nabla}\times\left[\frac{1}{\rho}\underline{\nabla} p\right] = \left(\theta_p \underline{\nabla} p + \theta_\rho \underline{\nabla}\rho\right)\cdot -\frac{1}{\rho^2}\underline{\nabla}\rho\times\underline{\nabla} p$

$= 0$. [following 2 pages do Ertel's theorem — for the full equations]

[if it really were $\underline{\nabla}\left(\frac{1}{\rho}\underline{\nabla} p\right)$, what do we even mean by that?]

~~it would have to be~~ $\cancel{\frac{1}{\rho}\nabla^2 p}$

1A

[Ertel's theorem]

Start with $\frac{d\rho}{dt} + \rho \nabla\cdot\underline{u} = 0$

$$\frac{d\underline{u}}{dt} + 2\underline{\Omega}\times\underline{u} = -\frac{1}{\rho}\nabla p - \nabla\Phi$$

$$\frac{d\theta}{dt} = 0 \qquad \Phi = gz$$

Define $\underline{\omega} = \text{curl}\,\underline{u}$, $\underline{\zeta} = \underline{\omega} + 2\underline{\Omega}$

$$\frac{\partial \underline{u}}{\partial t} + \nabla(\tfrac{1}{2}u^2) \underbrace{- \underline{u}\times\underline{\omega} + 2\underline{\Omega}\times\underline{u}}_{-\underline{u}\times\underline{\zeta}} = -\frac{1}{\rho}\nabla p - \nabla\Phi$$

curl: $$\frac{\partial\underline{\omega}}{\partial t} - \text{curl}(\underline{u}\times\underline{\zeta}) = -\text{curl}\left[\frac{1}{\rho}\nabla p\right]$$
$$= -\nabla\left(\frac{1}{\rho}\right)\times\nabla p$$

via $\text{curl}\,\phi\underline{a} = \phi\,\text{curl}\,\underline{a} + \nabla\phi\times\underline{a}$

note $\frac{\partial\underline{\omega}}{\partial t} = \frac{\partial\underline{\zeta}}{\partial t}$

$$\Rightarrow \nabla\theta\cdot\frac{\partial\underline{\zeta}}{\partial t} - \nabla\theta\cdot\text{curl}(\underline{u}\times\underline{\zeta}) = -\nabla\theta\cdot\left[\nabla\left(\frac{1}{\rho}\right)\times\nabla p\right] \qquad (*)$$

now $\nabla\cdot(\underline{a}\times\underline{b}) = \underline{b}\cdot\text{curl}\,\underline{a} - \underline{a}\cdot\text{curl}\,\underline{b}$

so $$-\underbrace{\nabla\theta}_{\underline{a}}\cdot\text{curl}(\underbrace{\underline{u}\times\underline{\zeta}}_{\underline{b}}) = \nabla\cdot\left[\nabla\theta\times(\underline{u}\times\underline{\zeta})\right] - (\underline{u}\times\underline{\zeta})\cdot\underbrace{\text{curl}\,\nabla\theta}_{=0}$$
$$= \nabla\cdot\left[\nabla\theta\times(\underline{u}\times\underline{\zeta})\right]$$

$\underline{a}\times(\underline{b}\times\underline{c}) = (\underline{a}\cdot\underline{c})\underline{b} - (\underline{a}\cdot\underline{b})\underline{c}$

Further $$\nabla\theta\times(\underline{u}\times\underline{\zeta}) = (\nabla\theta\cdot\underline{\zeta})\underline{u} - (\nabla\theta\cdot\underline{u})\underline{\zeta}$$

& since $\frac{d\theta}{dt} = 0$ this is $(\nabla\theta\cdot\underline{\zeta})\underline{u} + \underline{\zeta}\frac{\partial\theta}{\partial t}$

$\theta_t + \underline{u}\cdot\nabla\theta = 0$

So we have in (*)

$$\nabla\theta\cdot\frac{\partial \underset{\sim}{\xi}}{\partial t} + \nabla\cdot\left[(\nabla\theta\cdot\underset{\sim}{\xi})\underset{\sim}{u} + \underset{\sim}{\xi}\frac{\partial\theta}{\partial t}\right] = -\nabla\theta\cdot\left[\nabla\left(\tfrac{1}{\rho}\right)\times\nabla p\right]$$

$$\Rightarrow \nabla\theta\cdot\frac{\partial \underset{\sim}{\xi}}{\partial t} + \overbrace{\nabla\cdot\underset{\sim}{u}\,(\underset{\sim}{\xi}\cdot\nabla\theta) + \underset{\sim}{u}\cdot\nabla(\nabla\theta\cdot\underset{\sim}{\xi})}^{\nabla\cdot\phi\underset{\sim}{a} = \phi\nabla\cdot\underset{\sim}{a} + \underset{\sim}{a}\cdot\nabla\phi} + \underset{\sim}{\xi}\cdot\frac{\partial\nabla\theta}{\partial t} = -\nabla\theta\cdot\left[\nabla\left(\tfrac{1}{\rho}\right)\times\nabla p\right]$$

$$\frac{d}{dt}(\underset{\sim}{\xi}\cdot\nabla\theta) + \nabla\cdot\underset{\sim}{u}\,(\underset{\sim}{\xi}\cdot\nabla\theta) = -\nabla\theta\cdot\left[\nabla\left(\tfrac{1}{\rho}\right)\times\nabla p\right]$$

now $\nabla\cdot\underset{\sim}{u} = -\frac{1}{\rho}\frac{d\rho}{dt}$

$\div\rho$

$$\text{so}\quad \frac{1}{\rho}\frac{d}{dt}(\underset{\sim}{\xi}\cdot\nabla\theta) - \frac{1}{\rho^2}\frac{d\rho}{dt}(\underset{\sim}{\xi}\cdot\nabla\theta) = -\frac{1}{\rho}\nabla\theta\cdot\left[\nabla\left(\tfrac{1}{\rho}\right)\times\nabla p\right]$$

$$\text{ie}\quad \frac{d}{dt}\left(\frac{\underset{\sim}{\xi}\cdot\nabla\theta}{\rho}\right) = -\frac{1}{\rho}\nabla\theta\cdot\left[\nabla\left(\tfrac{1}{\rho}\right)\times\nabla p\right]$$

$$= \frac{1}{\rho^3}\nabla\theta\cdot\left[\nabla\rho\times\nabla p\right]$$

Note that if $\theta = \theta(p,\rho)$, so $\nabla\theta = \theta_p\nabla p + \theta_\rho\nabla\rho$

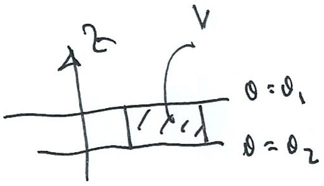
then RHS $= 0$.

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\underset{\sim}{\xi}\cdot\nabla\theta}{\rho}\right) = 0}$$

(iv) He doesn't define ζ !!

Now we're given $\frac{d}{dt}\left[\frac{(\zeta+f)\,\theta_z}{\rho}\right] = 0$

Here in fact ζ is the (vertical) vorticity $(v_x - u_y)$

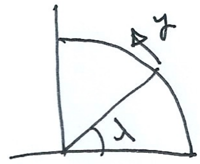


$f = 2\Omega\sin\lambda$

$\simeq \Omega\sqrt{2}$

if $\lambda = 45^\circ$ – (why !?)

– oh I suppose northern hemisphere



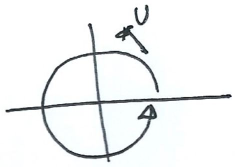
$\zeta + f = 0$ on V

moves north, f increases

so ζ decreases   Also $\zeta = -f$ so $\zeta < 0$

so the local rate of rotation decreases $-\zeta$ increases

[well $\zeta = v_x - u_y$



for rigid body anti-clockwise – i.e. cyclonic in northern hemisphere

$\zeta = \frac{U}{r} > 0$ so $\zeta < 0 \leftrightarrow$ ~~anticyclonic~~ anti-cyclonic (high pressure)

~~So a cyclonic (depression) low diminishes~~

$\Rightarrow$ ~~but an~~ anticyclonic (high) intensifies

[I suppose – seems a very poor question]