2009

Topics in fluids C5.7 [C6.4a] q3 answer (1

3/

$$\nabla \cdot \underline{u} = 0$$

$$\underline{u}_t + (\underline{u}\cdot\nabla)\underline{u} + \underline{f}\times\underline{u} = -\frac{1}{\rho_0}\nabla p + \frac{\rho}{\rho_0}\underline{g} + \frac{\mu}{\rho_0}\nabla^2\underline{u}$$

$$T_t + \underline{u}\cdot\nabla T = \kappa\nabla^2 T$$

$\underline{f} = f\underline{k}$ $\underline{g} = -g\underline{k}$

$\rho = \rho_0[1-\alpha(T-T_0)]$

(i) $x \sim d$, $t \sim \frac{d^2}{\kappa}$ $\underline{u} \sim \frac{\kappa}{d}$, $p + \rho_0 g z \sim \frac{\mu\kappa}{d^2}$, $T - T_0 \sim \Delta T$

$\Rightarrow$ non-d

$$\nabla\cdot\underline{u} = 0$$

$$\frac{\kappa}{d}\frac{\kappa}{d^2}\left[\frac{d\underline{u}}{dt}\right] + \frac{\kappa}{d}f\,\underline{k}\times\underline{u} = -\frac{\mu\kappa}{\rho_0 d^3}\nabla p + \alpha\Delta T g T\underline{k} + \frac{\mu}{\rho_0}\frac{\kappa}{d^3}\nabla^2\underline{u}$$

$\div$ by $\frac{\mu\kappa}{\rho_0 d^3}$

$$\Rightarrow \frac{\rho_0\kappa}{\mu}\frac{d\underline{u}}{dt} + \frac{\rho_0 d^2 f}{\mu}\underline{k}\times\underline{u} = -\nabla p + \frac{\alpha g\Delta T\rho_0 d^3}{\mu\kappa}T\underline{k} + \nabla^2\underline{u}$$

or $$\frac{1}{Pr}\frac{d\underline{u}}{dt} + Ta\,\underline{k}\times\underline{u} = -\nabla p + Ra\,T\underline{k} + \nabla^2\underline{u}$$

$Pr = \frac{\mu}{\rho_0\kappa}$ Prandtl $Ta = \frac{\rho_0 d^2 f}{\mu}$ Taylor $Ra = \frac{\alpha\rho_0 g\Delta T d^3}{\mu\kappa}$ Rayleigh

Energy equation is

$$T_t + \underline{u}\cdot\underline{\nabla} T = \nabla^2 T$$

(ii)

$$T = 1 - z$$

$$\underline{u} = 0$$

$$p_z = Ra(1-z)$$

$$\Rightarrow \quad p = -\tfrac{1}{2} Ra (1-z)^2$$

(iii) Small $\underline{u}$ linearise $p = -\frac{1}{2}Ra(1-z)^2 + P$, $T = 1 - z + \theta$

$$\Rightarrow \quad \frac{1}{Pr}\frac{\partial \underline{u}}{\partial t} + Ta\,\underline{k}\times\underline{u} = -\underline{\nabla} P + Ra\,\theta\underline{k} + \nabla^2 \underline{u}$$

$$\theta_t - w = \nabla^2\theta$$

$[\underline{u} = (u, v, w)]$ & $\underline{\nabla}\cdot\underline{u} = 0$

[To derive the dispersion relation...

Note that if $\underline{\nabla}\cdot\underline{u} = 0$ we can find $\underset{\sim}{\psi}$ s.t. $\underline{u} = \text{curl}\,\underset{\sim}{\psi}$

because solve $\nabla^2 \underset{\sim}{\Psi} = -\underset{\sim}{\omega} = -\text{curl}\,\underline{u}$ with appropriate bc's.

$\Rightarrow$ grad div $\underset{\sim}{\Psi}$ − curl curl $\underset{\sim}{\Psi}$ = − curl $\underline{u}$

Suppose div $\underset{\sim}{\Psi} = A$, then $\underline{\nabla} A = \text{curl}\,(\text{curl}\,\underset{\sim}{\Psi} - \underline{u})$

So $\nabla^2 A = 0$,

so solve $\nabla^2 \underset{\sim}{\Psi} = -\underset{\sim}{\omega}$ subject to $\underbrace{\underline{\nabla}\cdot\underset{\sim}{\Psi}}_{=A} = 0$ on boundary (eg at ∞)

$\Rightarrow A \equiv 0$

$\Rightarrow \underline{u} = \text{curl}\,\underset{\sim}{\Psi}$

etc.]

Assume the characteristic equation for solutions $\propto e^{\sigma t} e^{ikx} \sin n\pi z$

$$(\sigma + K_n^2)\left(\frac{\sigma}{Pr} + K_n^2\right)^2 K_n^2 - Ra\left(\frac{\sigma}{Pr} + K_n^2\right) K^2 + Ta(\sigma + K_n^2) n^2\pi^2 = 0$$

$$K_n^2 = k^2 + n^2\pi^2$$

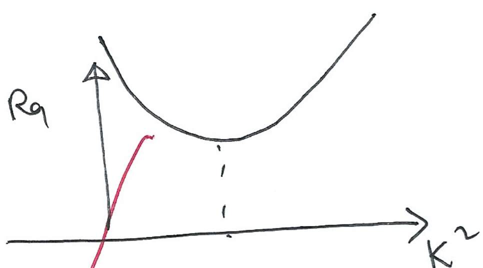
$$Ta = \frac{\rho_0 d^2 f}{\mu} \qquad (f = 2\Omega \text{ rotation})$$

If σ is real the critical Rayleigh number occurs when $\sigma = 0$

$$\Rightarrow \quad Ra = \cancel{K_n^2} \; \frac{K_n^6 + Ta\, n^2\pi^2}{K^2}$$

$$= \frac{(K^2 + n^2\pi^2)^3 + n^2\pi^2 Ta}{K^2}$$

or $Ta \to \infty$? if $Ta \gg 1$ $\quad Ra_c \simeq \dfrac{n^2\pi^2 Ta}{K^2}$ well no



note n=1 gives minimum

there is always a minimum

$$\frac{dRa}{dK^2} = \frac{3(K^2+n^2\pi^2)^2}{K^2} - \frac{\left\{(K^2+n^2\pi^2)^3 + n^2\pi^2 Ta\right\}}{K^4}$$

$$= 0 \text{ if } 3K^2(K^2+n^2\pi^2)^2 = (K^2+n^2\pi^2)^3 + n^2\pi^2 Ta$$

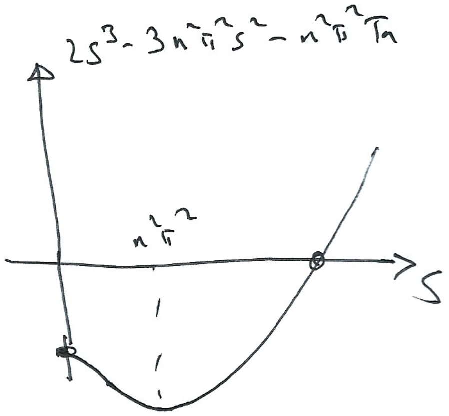
~~or~~ $\cancel{(K^2+n^2\pi^2)}$ $\cancel{K_n}$

write $K_n^2 = k^2 + n^2\pi^2 = s$

So $3(s - n^2\pi^2)\,s^2 = s^3 + n^2\pi^2 Ta$

ie $\cancel{3s^2}$ $2s^3 - 3n^2\pi^2 s^2 - n^2\pi^2 Ta = 0$

sketch



turning points at

$$6s^2 = 6n^2\pi^2 s$$

$$s = 0 \text{ or } n^2\pi^2$$

When Ta is large root is $s \simeq \left(\dfrac{n^2\pi^2 Ta}{2}\right)^{1/3}$

(other roots complex $s \simeq \pm i\left(\dfrac{Ta}{3}\right)^{1/2} + \dots$)

So min Ra is at $s \simeq \left(\dfrac{Ta\, n^2\pi^2}{2}\right)^{1/3}$

So $K \simeq \sqrt{s}$ (& take $n=1$)

$\Rightarrow K \simeq \pi^{1/3}\left(\dfrac{Ta}{2}\right)^{1/6}$

& $Ra_c \simeq \dfrac{K^6 + \pi^2 Ta}{K^2} \simeq \dfrac{\pi^2 \frac{Ta}{2} + \pi^2 Ta}{\left(\frac{Ta\,\pi^2}{2}\right)^{1/3}}$

$\simeq 3\left(\dfrac{\pi^2 Ta}{2}\right)^{2/3}$

rotation makes it more stable (as $Ra_c \uparrow Ta$).