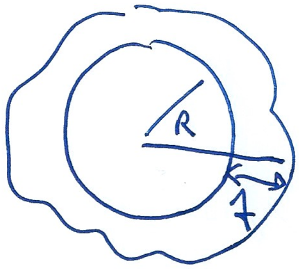
Topics in fluids 2010 q 2 answer

2)



$\cancel{Re\,\dot{u} = -p_r +}$

$$Re\left[u_t + uu_r + \frac{vu_\theta}{r} - \frac{v^2}{r}\right] = -p_r + u_{rr} + \frac{1}{r}u_r - \frac{u}{r^2} + \frac{u_{\theta\theta}}{r^2} - \frac{2v_\theta}{r^2} \qquad (1)$$

$$Re\left[v_t + uv_r + \frac{uv}{r} + \frac{vv_\theta}{r}\right] = -\frac{p_\theta}{r} + v_{rr} + \frac{v_r}{r} - \frac{v}{r^2} + \frac{v_{\theta\theta}}{r^2} + \frac{2u_\theta}{r^2} \qquad (2)$$

$$u_r + \frac{u}{r} + \frac{v_\theta}{r} = 0 \qquad (3)$$

on $r = R+\zeta$:

$$\zeta_t - u + \frac{v\zeta_\theta}{r} = 0 \qquad (4)$$

$$-p + \frac{2\left[r^2u_r - \zeta_\theta(u_\theta + rv_r - v) - u_r\zeta_\theta^2\right]}{r^2+\zeta_\theta^2} = \hat{C}\kappa \qquad (5)$$

$$-\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{r^2}{(r^2+\zeta_\theta^2)^{1/2}}\right] + \frac{1}{r}\frac{\partial}{\partial \theta}\left[\frac{\zeta_\theta}{(r^2+\zeta_\theta^2)^{1/2}}\right] = \kappa \qquad (6)$$

$$4\zeta_\theta r^2 u_r + (r^2-\zeta_\theta^2)(u_\theta + rv_r - v) = 0 \qquad (7)$$

Rescale $r = R+\delta\rho$, $\zeta = \delta h$

From (b) (note $\frac{\partial}{\partial r} = \frac{1}{\delta}\frac{\partial}{\partial \rho}$)

$$\kappa = -\frac{1}{R+\delta\rho}\frac{\partial}{\partial r}\left[\frac{(R+\delta\rho)^2}{\{(R+\delta\rho)^2 + \delta^2 h_\theta^2\}^{1/2}}\right]$$

$$+\frac{1}{R+\delta\rho}\frac{\partial}{\partial\theta}\left[\frac{\delta h_\theta}{\{(R+\delta\rho)^2 + \delta^2 h_\theta^2\}^{1/2}}\right]$$

$$\cancel{E \left[1 + \frac{\delta^2 h_\theta^2}{(R+\delta\rho)^2}\right]^{1/2} + \frac{1}{R+\delta\rho}}$$

$$= -\frac{1}{\delta(R+\delta\rho)}\frac{\partial}{\partial\rho}\left[(R+\delta\rho)\left\{1+\frac{\delta^2 h_\theta^2}{(R+\delta\rho)^2}\right\}^{-1/2}\right]$$

$$+\frac{1}{(R+\delta\rho)^2}\frac{\partial}{\partial\theta}\left[\delta h_\theta\left\{1+\frac{\delta^2 h_\theta^2}{(R+\delta\rho)^2}\right\}^{-1/2}\right]$$

$$= -\frac{1}{\delta(R+\delta\rho)}\frac{\partial}{\partial\rho}\left[(R+\delta\rho)\left\{1-\frac{\delta^2 h_\theta^2}{2(R+\delta\rho)^2} + O(\delta^4)\right\}\right]$$

$$+\frac{1}{(R+\delta\rho)^2}\frac{\partial}{\partial\theta}\left[\delta h_\theta + O(\delta^3)\right]$$

$$= -\frac{1}{(R+\delta\rho)}\left[1+O(\delta^2)\right] + \delta\frac{\partial}{\partial\rho}\left[\frac{h_\theta^2}{2(R+\delta\rho)^2} + O(\delta^2)\right]$$

$$+\frac{\delta h_{\theta\theta}}{(R+\delta\rho)^2} + O(\delta^3)$$

& to be evaluated on $\rho = h$, so

$$\kappa = \cancel{+\frac{1}{R+\delta\rho}} \; -\frac{1}{R+\delta h} + \cancel{\frac{\delta h_\theta^2}{2R^2}} \; \frac{\delta h_{\theta\theta}}{R^2} + O(\delta^2)$$

$$= -\frac{1}{R}\left(1-\frac{\delta h}{R}\cdots\right) + \frac{\delta h_{\theta\theta}}{R^2} \quad \cdots$$

$$\Rightarrow \kappa = -\frac{1}{R} + \frac{\delta}{R^2}\left(h_{\theta\theta} + h\right) \quad + O(\delta^2)$$

Rescale $\underline{u = \delta U}$, $t \sim O(1)$ (same)

$\left(\frac{\partial}{\partial r} = \frac{1}{\delta}\frac{\partial}{\partial \rho}\right)$

So we get

$$Re\left[\delta U_t + \delta U U_\rho + \frac{v u_\theta}{R+\delta\rho} - \frac{v^2}{R+\delta\rho}\right]$$

$$= -\frac{1}{\delta}p_\rho + \frac{1}{\delta}U_{\rho\rho} + \frac{1}{R+\delta\rho}U_\rho - \frac{\delta U}{(R+\delta\rho)^2} + \frac{\delta U_{\theta\theta}}{(R+\delta\rho)^2} - \frac{2v_\theta}{(R+\delta\rho)^2}$$

So at leading order (if $\delta^2 Re \ll 1$)

$$p_\rho = U_{\rho\rho} \qquad \text{(just wait...)}$$

and $$Re\left[v_t + U v_\rho + \frac{\delta U v}{R+\delta\rho} \cancel{\frac{v^2}{R+\delta\rho}} + \frac{v v_\theta}{R+\delta\rho}\right]$$

$$= -\frac{p_\theta}{R+\delta\rho} + \frac{1}{\delta^2}v_{\rho\rho} + \frac{1}{\delta}\frac{v_\rho}{R+\delta\rho} - \frac{v}{(R+\delta\rho)^2} + \frac{v_{\theta\theta}}{(R+\delta\rho)^2}$$

$$+ \frac{2\delta U_\theta}{(R+\delta\rho)^2}$$

So we need to rescale $p \sim \frac{1}{\delta^2}$

& then the r equation is $p_\rho \simeq 0 \Rightarrow p \simeq p(\theta, t)$

& if $\delta^2 Re \ll 1$

$$0 \simeq -\frac{p_\theta}{R} + v_{\rho\rho}$$

We have $u = v = 0$ on $\rho = 0$

& on $\rho = h$:
$$\delta h_t - \delta U + \frac{\delta v h_\theta}{R + \delta\rho} = 0$$

$$\Rightarrow h_t - U + \frac{v h_\theta}{R} \simeq 0$$

with rescale p : on $\rho = h$

$$-\frac{p}{\delta^2} + \frac{2\left[\left[(R+\delta\rho)^2 U_\rho - \delta h_\theta \left\{ \delta U_\theta + \frac{1}{\delta}(R+\delta\rho) v_\rho - v \right\} - \delta^2 U_\rho h_\theta^2\right]}{(R+\delta\rho)^2 + \delta^2 h_\theta^2} = \hat{C}\kappa$$

So
$$-p + O(\delta^2) = \hat{C}\kappa$$
$$= \delta^2 \hat{C}\left[-\frac{1}{R} + \frac{\delta}{R^2}(h_{\theta\theta} + h) + O(\delta^2)\right] (\delta)$$

and
$$4\delta h_\theta (R+\delta\rho)^2 U_\rho + \left\{(R+\delta\rho)^2 - \delta^2 h_\theta^2\right\}\left[\delta U_\theta + \frac{1}{\delta}(R+\delta\rho) v_\rho - v\right] = 0$$

$$\Rightarrow v_\rho = O(\delta) \quad \text{on } \rho = h$$

At leading order $\quad v_{\rho\rho} = \dfrac{p_\theta}{R}$

$$v_\rho = -\frac{p_\theta}{R}(h-\rho)$$

$$v = -\frac{p_\theta}{R}\left(h\rho - \tfrac{1}{2}\rho^2\right)$$

and on $\rho = h$: $\quad \underline{h_t - U + \dfrac{v h_\theta}{R} \simeq 0} \qquad (*)$

& mass is $\quad U_\rho + \dfrac{\delta U}{R+\delta\rho} + \dfrac{v_\theta}{R+\delta\rho} = 0$

at lead order $\quad \underline{U_\rho + \dfrac{1}{R} v_\theta = 0} \qquad (\dagger)$

So $\quad h_t = U|_h - \dfrac{v|_h h_\theta}{R}$

Note we define $\quad \bar{v} = \dfrac{1}{\delta}\int_0^f v\,dr = \dfrac{1}{h}\int_0^h v\,d\rho$

so $h\bar{v} = \displaystyle\int_0^h v\,d\rho = -\frac{p_\theta}{R}\left[\tfrac{1}{2}h\rho^2 - \tfrac{1}{6}\rho^3\right]_0^h$

$$= -\frac{1}{3}h^3\frac{p_\theta}{R}$$

and note $\quad \dfrac{\partial}{\partial\theta}\displaystyle\int_0^h v\,d\rho = v|_h h_\theta + \int_0^h v_\theta\,d\rho$

(use $(*)$) $\quad = R(U|_h - h_t) - \displaystyle\int_0^h R U_\rho\,d\rho$

& $(\dagger)$ $\quad = -Rh_t + RU|_h - RU|_0^h$

$$= -Rh_t$$

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Thus $\quad R h_t + \dfrac{\partial}{\partial\theta}\displaystyle\int_0^h v\,d\rho = 0$

or $\quad h_t + \dfrac{1}{R}\dfrac{\partial}{\partial\theta}(h\bar{v}) = 0$

$$h\bar{v} = -\tfrac{1}{3}h^3\,\frac{p_\theta}{R}$$

And p is given by $\quad p = \delta^2\hat{C}\left[\dfrac{1}{R} - \dfrac{\delta}{R^2}(h_{\theta\theta}+h)\ldots\right]$ (δ) on p4)

~~also~~ $\Rightarrow \quad -\dfrac{p_\theta}{R} = \dfrac{\delta^3\hat{C}}{R^3}(h_{\theta\theta}+h)$

choose $\quad \underline{\delta^3\hat{C} = C}$

$$\underline{\Rightarrow \quad h_t + \frac{1}{R}\frac{\partial}{\partial\theta}\left[\frac{1}{3}h^3\,\frac{C}{R^3}(h_{\theta\theta}+h)\right] = 0}$$

as required

uniform film $h = h_0$ is a steady solution

put $h = h_0 + H$, linearise

$$\Rightarrow \quad H_t + \frac{C}{3R^4}\frac{\partial}{\partial\theta}\left[\tfrac{1}{3}(h_0^3 + 3h_0^2H\ldots)(h_0 + H + H_{\theta\theta})\right] = 0$$

ie $\quad H_t + \dfrac{C}{3R^4}\dfrac{\partial}{\partial\theta}\left[h_0^3H + \tfrac{1}{3}h_0^3(H_{\theta\theta}+H)\right] = 0$

$H = e^{\sigma t + ik\theta}$

$$\Rightarrow \quad \sigma + \frac{Ch_0^3}{3R^4}\left[\tfrac{4}{3}ik + \tfrac{1}{3}i^3k^3\right] = 0$$

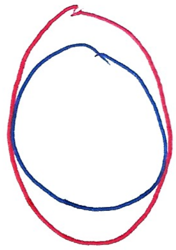
$$-\frac{\sigma}{ik} = \frac{Ch_0^3}{9R^4}(4-k^2)$$

~~neutrally~~ stable waves with speed $-\dfrac{\sigma}{ik}$. $\quad \sigma = 0 \quad k = \pm 2$

$\leftrightarrow$ H or $e^{\pm 2i\theta}$

then neutrally stable (as $\omega_i = \sigma = 0$) for $H = e^{\pm 2i\theta}$ (7)

i.e stationary waves



e.g.

I think this is what is required

but I'm not sure why they are

stationary

(remember no gravity)